

Seifert Fibered Spaces

Notes for a course given in the Spring of 1993

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As an exercise in learning L^AT_EX, these notes were reset in January of 1996.

The appearance has changed, but the content remains the same.

Preface to the arXiv edition

These notes have resided on my web page for as long as I have had a web page. They have been referred to at least once in a published article. It has taken me this long (1993–2007) to figure out that it would make sense for them to reside in the arXiv instead. I don't know that the arXiv has been in existence since 1993, but it has been around long enough for me to know better. From now on my web page will have a link to the copy in the arXiv and I will get out of the business of serving these notes to the public.

The notes are not complete. They contain a description of compact three dimensional Seifert fibered spaces and a classification up to homeomorphism of compact three dimensional Seifert fibered spaces with non-empty boundary. It would have been nicer to go all the way to a classification of all compact three dimensional Seifert fibered spaces both with and without boundary, but time ran out and I never added to the notes beyond what was covered in the course.

Binghamton, 2007

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CHAPTER 1

Basics

1.1. Introduction

Most of this chapter is taken from the first 8 of the 15 sections of Seifert's original paper on fibered spaces: H. Seifert, "Topologie dreidimensionaler gefaserte Räume," Acta Math. 60 (1932), 147–238. An English translation by W. Heil: "Topology of 3-dimensional fibered spaces," appears starting on page 359 of "Seifert and Threlfall: A textbook in topology" (yes that is the full title) by (of course) H. Seifert and W. Threlfall, an English translation by M. Goldman of Seifert and Threlfall's 1934 book "Lehrbuch der Topologie." The English volume in question is edited by J. Birman and J. Eisner and was published in 1980 by Academic Press, New York, as volume 89 in the series on Pure and Applied Mathematics. The examples in this chapter were mostly taken from the book by W. Jaco that is identified more completely in the introductory paragraph to Chapter 2.

Seifert fibered spaces are 3-manifolds that are unions of pairwise disjoint circles. There will be restrictions on how the circles fit together. These restrictions will be discussed shortly. If M is a Seifert fibered space, then the circles just referred to are the "fibers" of M . If M and N are Seifert fibered spaces, then a "fiber preserving homeomorphism" from M to N is a homeomorphism from M to N that takes each fiber of M onto a fiber of N . If such a homeomorphism exists, then we say that M and N are "fiber homeomorphic." Our first tasks

will be to define Seifert fibered spaces, to classify Seifert fibered spaces up to fiber preserving homeomorphism, and then to classify Seifert fibered spaces up to arbitrary homeomorphism. [The course ended before this last part could be accomplished. Compact Seifert fibered spaces with non-empty boundary were classified.] One expects the last classification to be somewhat looser than the first since more homeomorphisms are allowed. For the most part this is not the case, and the situations in which classes combine because of the extra homeomorphisms are few in number and easy to describe. To save words, we define the "*fiber type*" of a Seifert fibered space to be the class of all Seifert fiber spaces that are fiber homeomorphic to it.

1.2. Definition of Seifert fibered spaces

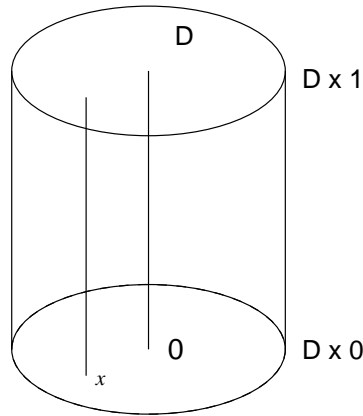
Many spaces are defined using local models. An n -manifold without boundary is defined to be a separable metric space in which every point has an open neighborhood homeomorphic to the fixed local model \mathbf{R}^n . An n -manifold with boundary is a separable metric space in which every point has a closed neighborhood homeomorphic to the fixed local model \mathbf{B}^n . Seifert fibered spaces are defined using an infinite collection of local models, and these models are not models for neighborhoods of points but for neighborhoods of fibers.

1.2.1. Structure of the local model. Let D be the unit disk in the complex plane \mathbf{C} and let $D \times I$ be fibered by the intervals $\{x\} \times I, x \in D$ as shown in Figure 1.1.

Let ν and μ be integers with $\mu \neq 0$. Define $\rho : D \rightarrow D$ by

$$\rho(x) = xe^{2\pi i(\nu/\mu)}.$$

That is, ρ rotates D by ν/μ of a full circle. The homeomorphism ρ is completely determined by the rational number $\nu/\mu \pmod{1}$. Thus we assume that ν and μ

FIGURE 1.1. $D \times I$

are relatively prime, that $\mu > 0$ and that $0 \leq \nu < \mu$. Let T be the quotient of $D \times I$ obtained by identifying $(x, 0)$ with $(\rho(x), 1)$ for each $x \in D$. Since ρ is isotopic to the identity map, T is homeomorphic to $D \times S^1$.

For a given $x \in D$, there are a finite number of images of x in D under powers of ρ . Since ν and μ are relatively prime, this number of images is exactly μ if x is not the center of D , and there is exactly one image if $x = 0$, the center of D . We can now describe T as a union of circles. One circle is the image of $\{0\} \times I$ under the quotient map. The other circles are each the union of images of μ intervals of the form $\{\rho^i(x)\} \times I$ where i runs from 0 to $\mu - 1$. We call these circles the "fibers" of T . We use T to refer to both the underlying solid torus together with the added structure consisting of the set of fibers of T . We call such a T a "fibered solid torus". We call the fiber of T that is the image of $\{0\} \times I$ the "centerline" of T . We refer to T as the fibered solid torus determined by the rational number $\nu/\mu \pmod{1}$.

If $\nu = 0$, then the fibering of T is the product fibering of $D \times S^1$. In this case we call T an "ordinary solid torus".

There are many ordinary solid tori in a fibered solid torus. Let x be a point of D other than the center and let T be the fibered solid torus determined by the rational number ν/μ as a quotient of $D \times I$. Let H be the fiber of T containing the image of $\{x\} \times I$. There is a disk E in D that contains a neighborhood x and does not contain the center of D . If x is in ∂D , then x is in ∂E , and if x is in $\overset{\circ}{D}$, then x is in $\overset{\circ}{E}$. By choosing E small enough, we can keep E disjoint from its μ images under powers of ρ . Now the union of the images of $\rho^i(E) \times I$, $0 \leq i < \mu$ is a solid torus T' in T fibered as an ordinary solid torus by the fibers of T that it contains. If x is in $\overset{\circ}{E}$, then we can regard x as the center point of E . Thus H can be regarded as the centerline of T' . So every fiber in T that is not the centerline of T is contained in a union of fibers of T that forms an ordinary solid torus in T . Further, if the fiber is in $\overset{\circ}{T}$, then it is the centerline of this ordinary solid torus.

A fibered solid torus T is orientable. If $D \times I$ is given an orientation, then T can inherit the orientation from $D \times I$. We will show below that the choice of orientation is important in determining certain invariants. We will therefore choose consistent orientations for the various local models. We will do this by picking one orientation for $D \times I$ and letting the models inherit that orientation. When we discuss below invariants of fibered solid tori, we will then describe our choice of orientation for $D \times I$.

If T is the fibered solid torus determined by ν/μ as a quotient of $D \times I$, then there is a fiber preserving homeomorphism from T to the image of $D' \times I$ where D' is the half unit disk in \mathbf{C} . The homeomorphism is induced by radial dilation by one half.

1.2.2. The definition. Let a 3-manifold M be a pairwise disjoint union of simple closed curves called the "fibers" of M . We will use M to refer to the underlying 3-manifold together with the added structure consisting of the set of

fibers of M . A subspace of M is "saturated" if it is a union of fibers. Such a subspace is given the extra structure of its set of fibers. We say that M is a "Seifert fibered space" if each fiber H of M has a saturated, closed neighborhood N with a fiber preserving homeomorphism from N to a fibered solid torus T . If M is oriented, then we require that the fiber preserving homeomorphism from N to T be orientation preserving.

We pick out certain saturated torus neighborhoods of fibers for special consideration. If a fiber H has a point in $\overset{\circ}{M}$, then $H \subseteq \overset{\circ}{N}$ and $H \subseteq \overset{\circ}{M}$. Thus a fiber of M lies either entirely in ∂M or entirely in $\overset{\circ}{M}$. If H lies in ∂M , then a saturated neighborhood N of H that is fiber homeomorphic to a fibered solid torus must have H in ∂N . By previous remarks, a possibly smaller saturated neighborhood of H will be fiber homeomorphic to an ordinary solid torus. That this might not be all of N is seen by letting M be a non-ordinary fibered solid torus and letting $N = M$.

If H lies in $\overset{\circ}{M}$, then there is a saturated neighborhood N of H that has a homeomorphism to a fibered solid torus T . The homeomorphism carries H into $\overset{\circ}{T}$, and may or may not carry H to the centerline of T . If H is not carried to the centerline of T , then previous remarks imply that there is a smaller saturated neighborhood of H that is fiber homeomorphic to an ordinary solid torus T' under a homeomorphism that carries H to the centerline of T' . By narrowing the neighborhood, we can preserve all that we have and insist that the neighborhood lie in $\overset{\circ}{M}$.

Thus every fiber in M is either in ∂M and contained in the boundary of an ordinary solid torus, or is in $\overset{\circ}{M}$ and is the centerline of a fibered (perhaps ordinary) solid torus in $\overset{\circ}{M}$. For a fiber H in M , we say that N is a "fibered solid torus neighborhood" of H if H is in ∂M and N is a saturated neighborhood of H that is fiber homeomorphic to an ordinary solid torus, or if H is in $\overset{\circ}{M}$ and

N lies in $\overset{\circ}{M}$ and has a fiber preserving homeomorphism to a fibered solid torus T that carries H to the centerline of T . We will see later that H determines its fibered solid torus neighborhoods in a strong way.

Since a fiber of M that intersects that boundary of M must lie in the boundary of M and have an orientable neighborhood in ∂M , it follows that boundary components of M are saturated tori, Klein bottles and open annuli. Klein bottles can arise because it is possible to represent a Klein bottle as a union of annuli.

1.3. Fibered solid tori

Before we look at Seifert fibered spaces, it will be useful to study the local model.

1.3.1. Background. We need some information about solid tori and their boundary tori. We will take the facts in this item as given. Let T be a solid torus and let ∂T be its boundary torus. Both $\pi_1(T)$ and $H_1(T)$ are isomorphic to \mathbf{Z} and both $\pi_1(\partial T)$ and $H_1(\partial T)$ are isomorphic to $\mathbf{Z} \oplus \mathbf{Z}$.

We concentrate for a while on $H_1(\partial T)$. Once an ordered pair of generators is chosen for $H_1(\partial T)$, we can think of elements of $H_1(\partial T)$ as column vectors $\begin{pmatrix} i \\ j \end{pmatrix}$ with integer entries where i is the coefficient of the first generator and j is the coefficient of the second. Automorphisms of $H_1(\partial T)$ are then represented by two by two integer matrices $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ with $\begin{vmatrix} a & b \\ c & d \end{vmatrix} = \pm 1$. If a different pair of generators of $H_1(\partial T)$ is chosen, then the matrix representing an automorphism will change, but its determinant will not. Thus the determinant of an automorphism of $H_1(\partial T)$ is well defined.

Given a pair of generators (\mathbf{u}, \mathbf{v}) of $H_1(\partial T)$, there is a unique anti-symmetric pairing (bilinear form) $\psi(\cdot, \cdot)$ on $H_1(\partial T)$ for which $\psi(\mathbf{u}, \mathbf{v}) = +1$. It is defined by $\psi\left(\begin{pmatrix} a \\ b \end{pmatrix}, \begin{pmatrix} c \\ d \end{pmatrix}\right) = \begin{vmatrix} a & c \\ b & d \end{vmatrix}$. The representing matrix of the pairing is $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. For an element $\mathbf{w} = a\mathbf{u} + b\mathbf{v}$ of $H_1(\partial T)$, we have $\psi(\mathbf{w}, \mathbf{u}) = -b$, $\psi(\mathbf{w}, \mathbf{v}) = a$, and

$\mathbf{w} = \psi(\mathbf{w}, \mathbf{v})\mathbf{u} - \psi(\mathbf{w}, \mathbf{u})\mathbf{v} = \psi(\mathbf{w}, \mathbf{v})\mathbf{u} + \psi(\mathbf{u}, \mathbf{w})\mathbf{v}$. This is useful since it allows coefficients to be read off from the pairing. The pairing is preserved by automorphisms of determinant $+1$ and negated by automorphisms of determinant -1 . This pairing and its negative are the only two anti-symmetric pairings on $H_1(\partial T)$ whose image is all of \mathbf{Z} . We can think of a choice of orientation of ∂T either as a choice of one of the two classes of ordered pairs of generators of $H_1(\partial T)$ connected by automorphisms of determinant $+1$, or as a choice of one of the two anti-symmetric pairings on $H_1(\partial T)$ whose image is all of \mathbf{Z} . A choice (\mathbf{u}, \mathbf{v}) of generating pair and a choice of pairing $\psi(\cdot, \cdot)$ is consistent if $\psi(\mathbf{u}, \mathbf{v}) = +1$. We can abuse notation and think of this as a choice of orientation for $H_1(\partial T)$ as well and label automorphisms of $H_1(\partial T)$ as orientation preserving or orientation reversing as given by the determinant.

Let a pair of generators (\mathbf{u}, \mathbf{v}) of $H_1(\partial T)$ be given and let $\psi(\cdot, \cdot)$ be the anti-symmetric pairing on $H_1(\partial T)$ for which $\psi(\mathbf{u}, \mathbf{v}) = +1$. Assume that

$$\psi\left(\begin{pmatrix} a \\ b \end{pmatrix}, \begin{pmatrix} c \\ d \end{pmatrix}\right) = \epsilon = \pm 1$$

for elements $\begin{pmatrix} a \\ b \end{pmatrix}$ and $\begin{pmatrix} c \\ d \end{pmatrix}$ in $H_1(\partial T)$. Then the matrix $\begin{pmatrix} a & c \\ b & d \end{pmatrix}$ represents an automorphism of $H_1(\partial T)$ which preserves orientation depending on ϵ and which carries \mathbf{u} to $\begin{pmatrix} a \\ b \end{pmatrix}$ and \mathbf{v} to $\begin{pmatrix} c \\ d \end{pmatrix}$. Thus $\left(\begin{pmatrix} a \\ b \end{pmatrix}, \begin{pmatrix} c \\ d \end{pmatrix}\right)$ is a generating pair for $H_1(\partial T)$ and gives the same orientation as (\mathbf{u}, \mathbf{v}) if and only if $\epsilon = +1$. Thus the pairing picks out generating pairs for $H_1(\partial T)$.

Another view of an orientation for ∂T is that it gives a consistent way of assigning one of ± 1 to an isolated, transverse intersection of an ordered pair of oriented open arcs in ∂T . This can be extended by summing to a definition of intersection number for ordered pairs of oriented closed curves that can be shown to depend only on the homology classes that the curves represent and to be bilinear. This then gives a well defined notion of intersection numbers on ordered pairs of homology classes in $H_1(\partial T)$. By considering the “obvious”

generators of $H_1(\partial T)$ — two coordinate slices of $S^1 \times S^1$ — it can be seen that the intersection numbers give an anti-symmetric pairing whose image is all of \mathbf{Z} . Thus the pairing is one of the “orientation pairings” discussed above. We can now refer to the “orientation pairings” as intersection pairings.

We can use the intersection pairing to gather information about simple closed curves on ∂T . A simple closed curve on ∂T separates if and only if it is null homologous in ∂T . If a simple closed curve J represents a non-trivial element of $H_1(\partial T)$, then it does not separate, and it is transverse to some simple closed curve K that it intersects exactly once. Thus $\psi(J, K) = \pm 1$ and (J, K) represents a generating pair for $H_1(\partial T)$. Further, the homology class $[J]$ that J represents is indivisible in $H_1(\partial T)$ — it is an integer multiple of no element other than itself or its inverse — and if $[J] = \begin{pmatrix} a \\ b \end{pmatrix}$ with respect to an arbitrary pair of generators for $H_1(\partial T)$, then a and b are relatively prime. A converse is true. If a class in $H_1(\partial T)$ is indivisible, then it is represented by a simple closed curve. This can be seen by using the “obvious” generating pair, noting that the coefficients of the element must be relatively prime, and then plotting a lift of the desired curve in the universal cover as a straight line with slope the quotient of the coefficients.

We need more information about curves. (Is a good reference for this paragraph the paper by Epstein on curves on surfaces and isotopies?) If two simple closed curves on ∂T have intersection pairing n , then one curve can be altered by an isotopy so that the two curves intersect transversely and intersect in exactly $|n|$ points. From this it follows that every automorphism of $H_1(\partial T)$ is realized by some homeomorphism of ∂T because the new generators can now be represented by a pair of transverse simple closed curves that intersect in one point and it is easy to construct a homeomorphism carrying the “obvious” generators to the new generating curves. A second fact classifies homeomorphisms. Two simple closed curves on ∂T are homologous if and only if they are homotopic in ∂T , in

which case they are also isotopic in ∂T . (The last part generalizes to all surfaces. Two embedded curves are homotopic if and only if they are isotopic where the embeddings, homotopies and isotopies are all required to preserve boundaries in that inverse images of boundaries are always boundaries.) From this information and the Alexander Lemma it can be shown that a homeomorphism that induces the identity on $H_1(\partial T)$ is isotopic to the identity on ∂T . Thus self homeomorphisms of ∂T are classified up to isotopy by the induced automorphisms on $H_1(\partial T)$ and, once an ordered pair of generators for $H_1(\partial T)$ is chosen, by two by two integer matrices of determinant ± 1 . Choosing ordered pairs of generators for the first homologies of a pair of tori allows the classification of homeomorphisms, up to isotopy, between the tori by two by two, invertible, integer matrices.

We call a simple closed curve on ∂T that is null homotopic in T but not in ∂T a "*meridian*" of T . There are only two classes in $H_1(\partial T)$ that contain meridians of T . These classes are the negatives of each other so that there is only one meridian of T up to isotopy and reversal of orientation. A self homeomorphism of ∂T extends to all of T if and only if the homology class of the meridian is preserved up to sign. We call any simple closed curve ℓ on ∂T a "*longitude*" for T if there is a meridian for T that intersects ℓ exactly once and which is transverse to ℓ . A longitude for T represents a generator of $\pi_1(T) = H_1(T) = \mathbf{Z}$ and a meridian-longitude pair represents a generating pair for $H_1(\partial T)$. Any meridian-longitude pair for T can be carried to any other meridian-longitude pair for T by a self homeomorphism of T .

We will be interested in understanding fiber preserving homeomorphisms of fibered solid tori. We will use this information to obtain numerical invariants that classify fibered solid tori up to fiber preserving homeomorphism, and to obtain geometric invariants that allow us to unambiguously specify sewings of fibered solid tori up to fiber preserving deformations. We will see that all of

the information that we need is contained in the boundaries of the fibered solid tori, and that in the boundaries, it is the homology classes of curves that are important. Understanding how classes of curves in a torus behave under certain homeomorphisms of the torus needs an understanding of how subgroups of $\mathbf{Z} \oplus \mathbf{Z}$ behave under certain automorphisms of $\mathbf{Z} \oplus \mathbf{Z}$.

1.3.2. Automorphisms of $\mathbf{Z} \oplus \mathbf{Z}$. Let T be a fibered solid torus. Let H be a fiber of T on ∂T , and let m be a meridian for T . Fiber preserving self homeomorphisms of T will take H to H or its reverse and m to m or its reverse. The induced automorphism on $H_1(\partial T)$ will preserve the cyclic subgroups generated by the classes of H and of m . These cyclic subgroups are maximal cyclic. They are also different since $\psi(H, m)$ is never 0. This motivates what we look at in this section.

Let $G = \mathbf{Z} \oplus \mathbf{Z}$. We identify G with column vectors $\begin{pmatrix} i \\ j \end{pmatrix}$. We use the notation (i, j) for the greatest common divisor of i and j . A cyclic subgroup of G is generated by some $\begin{pmatrix} i \\ j \end{pmatrix}$ in G other than the zero element $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$. This subgroup will be maximal cyclic in G if and only if $(i, j) = 1$. Every non-zero element of G is contained in a unique maximal cyclic subgroup of G . The element $\begin{pmatrix} i \\ j \end{pmatrix}$ is contained in the maximal cyclic subgroup generated by

$$\begin{pmatrix} \frac{i}{(i, j)} \\ \frac{j}{(i, j)} \end{pmatrix}.$$

We use the convention that $(i, 0) = i$ if $i \neq 0$. It follows that $\begin{pmatrix} i \\ j \end{pmatrix}$ and $\begin{pmatrix} i' \\ j' \end{pmatrix}$ are contained in the same maximal cyclic subgroup if and only if $i/j = i'/j'$ where all fractions with denominator 0 are taken to be the same. We can thus use fractional notation to identify maximal cyclic subgroups of G , and we use $[i/j]$ to refer to the unique maximal cyclic subgroup of G containing $\begin{pmatrix} i \\ j \end{pmatrix}$.

Let ϕ be an automorphism of G . We identify ϕ with the 2×2 integer matrix that represents ϕ . We can thus talk about $\det(\phi)$ and we know that $\det(\phi) = \pm 1$. We say that ϕ is "orientation preserving" if $\det(\phi) = 1$ and "orientation reversing" if $\det(\phi) = -1$.

In the following, we consider automorphisms that fix $[1/0]$ because homeomorphisms of the boundary of a solid torus must preserve the meridian if they are to extend to the entire solid torus.

LEMMA 1.3.1. *Let $[\nu/\mu]$ and $[\nu'/\mu']$ be maximal cyclic subgroups of G with $\mu \neq 0$ and $\mu' \neq 0$. There is an orientation preserving automorphism ϕ of G fixing $[1/0]$ and taking $[\nu/\mu]$ to $[\nu'/\mu']$ if and only if $\nu/\mu \equiv \nu'/\mu' \pmod{1}$. There is an orientation reversing automorphism ϕ of G fixing $[1/0]$ and taking $[\nu/\mu]$ to $[\nu'/\mu']$ if and only if $\nu/\mu \equiv -\nu'/\mu' \pmod{1}$.*

PROOF Assume a ϕ exists carrying $[\nu/\mu]$ to $[\nu'/\mu']$. Because ϕ preserves $[1/0]$, it must have the form:

$$\epsilon \begin{pmatrix} \omega & b \\ 0 & 1 \end{pmatrix}$$

where $\epsilon = \pm 1$ and $\omega = \pm 1$. Here $\omega = \det(\phi)$ and determines whether or not ϕ preserves orientation. We assume that ν/μ and ν'/μ' have been reduced to lowest terms so that $\begin{pmatrix} \nu \\ \mu \end{pmatrix}$ and $\begin{pmatrix} \nu' \\ \mu' \end{pmatrix}$ are generators of the respective maximal cyclic subgroups. We have

$$\pm \begin{pmatrix} \nu' \\ \mu' \end{pmatrix} = \epsilon \begin{pmatrix} \omega & b \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \nu \\ \mu \end{pmatrix} = \epsilon \begin{pmatrix} \omega\nu + b\mu \\ \mu \end{pmatrix}.$$

This immediately gives that $\nu'/\mu' \equiv \omega\nu/\mu \pmod{1}$. The converse is a straightforward reversal of the argument. ■

From the above lemma, the orbit of $[\nu/\mu]$ under orientation preserving automorphisms of G that preserve $[1/0]$ can be represented by the fraction ν/μ with $(\nu, \mu) = 1$, $\mu > 0$ and $0 \leq \nu < \mu$. This is the reason for the restriction in the next lemma.

LEMMA 1.3.2. *Let $[\nu/\mu]$ be a maximal cyclic subgroup of G with $(\nu, \mu) = 1$, $\mu > 0$ and $0 \leq \nu < \mu$. The only automorphisms of G that preserve each of $[\nu/\mu]$ and $[1/0]$ are among*

$$\phi_1 = \pm \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \phi_2 = \pm \begin{pmatrix} -1 & 1 \\ 0 & 1 \end{pmatrix}, \text{ and } \phi_3 = \pm \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Only ϕ_1 works for all $[\nu/\mu]$, ϕ_2 works only for $[1/2]$, and ϕ_3 works only for $[0/1]$.

PROOF One can check that the given automorphisms work as stated. We show there are no more. From the proof of the lemma above, we have

$$\pm \begin{pmatrix} \nu \\ \mu \end{pmatrix} = \epsilon \begin{pmatrix} \omega\nu + b\mu \\ \mu \end{pmatrix}$$

so $\nu = \omega\nu + b\mu$. If $\omega = 1$, then $b = 0$. If $\omega = -1$, then $2\nu = b\mu$ and either $b = \nu = 0$ or $\nu \mid \mu$ since $0 \leq \nu < \mu$. So if $b \neq 0$, then $b = 1$, $\nu = 1$ and $\mu = 2$. ■

COROLLARY 1.3.2.1. *Let $[\nu/\mu]$ be a maximal cyclic subgroup of G with $\mu > 0$ and $(\nu, \mu) = 1$. The only automorphisms of G that preserve each of $[\nu/\mu]$ and $[1/0]$ are among*

$$\phi_1 = \pm \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \phi_2 = \pm \begin{pmatrix} -1 & \nu \\ 0 & 1 \end{pmatrix}, \text{ and } \phi_3 = \pm \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Only ϕ_1 works for all $[\nu/\mu]$, ϕ_2 works only if $\mu = 2$, and ϕ_3 works only if $\nu = 0$.

PROOF This is obtained by conjugating the automorphisms of the previous lemma by an automorphism that takes $[\nu/\mu]$ to $[\nu'/\mu]$ with $0 \leq \nu' < \mu$. ■

[I am indebted to Yuncherl Choi for pointing out an omission of the automorphism ϕ_3 in the last two lemmas.]

The lemmas above apply to isomorphisms between two copies of $\mathbf{Z} \oplus \mathbf{Z}$ once a fixed ordered pair of generators is chosen for each copy.

1.3.3. Homeomorphisms of fibered solid tori. We start by showing that the automorphisms of $\mathbf{Z} \oplus \mathbf{Z}$ identified above can be realized by fiber preserving homeomorphisms of a fibered solid torus T . We need certain conventions first.

Let m be a meridian of T and let ℓ be a longitude for T . We identify $H_1(\partial T)$ with $\mathbf{Z} \oplus \mathbf{Z}$ and use m to represent the generator of the first factor and use ℓ to represent the generator of the second factor. Elements of $H_1(\partial T)$ are column vectors with the first (top) component giving the coefficient of m . A self homeomorphism of ∂T , induces an automorphism of $H_1(\partial T)$ represented by a 2×2 integer matrix with determinant ± 1 that acts on elements of $H_1(\partial T)$ on the left. The choice of (m, ℓ) as the generating pair for $H_1(\partial T)$ determines the orientation of ∂T and the intersection pairing on $H_1(\partial T)$. Recall that this pairing is anti-symmetric and has $\psi(m, \ell) = 1$.

Longitudes of a solid torus are not unique. There are also two choices for the orientation of the meridian. We now make choices for our convenience by referring to $D \times I$.

Let T be a fibered solid torus obtained from $D \times I$ using the homeomorphism ρ determined by ν/μ with ν relatively prime to μ . Let ℓ be the image in T of the curve

$$L(t) = (e^{2\pi i(\nu t/\mu)}, t)$$

in $D \times I$. The ends of L are $(1, 0)$ and $(\rho(1), 1)$ so that ℓ is a simple closed curve in T and in fact a longitude of T transverse to the meridian m which is the image of $\partial D \times \{0\}$. We now pick orientations. We orient ∂D counterclockwise, and we orient L from $(\rho(1), 1)$ to $(1, 0)$. By declaring that ∂D comes “before” L , we determine an orientation of $\partial(D \times I)$ and thus on $D \times I$. Orientations are now inherited by $m, \ell, \partial T$ and T . We make these choices because they cooperate well with the fibers of T . Let H be a fiber on ∂T . If H is oriented by orienting the

fibers $\{x\} \times I$ in $D \times I$ from the 1 level towards the 0 level, then $\psi(H, m) = -\mu$ and $\psi(H, l) = \nu$. Thus H represents $\nu m + \mu l$ in $H_1(\partial T)$. This recovers ν/μ as the ratio of the coefficients of m and l . The choice of orientation for H is actually not important since reversing the orientation of H has $H = -\nu m - \mu l$ and ν/μ is still recovered in the same way.

Note that $\psi(H, m)$ is never 0. Thus H and m are never homologous in ∂T and H is never null homologous in T .

We now describe three self homeomorphisms of $D \times I$. The first h_1 is the identity. The second is defined by $h_2(x, t) = (\bar{x}, 1 - t)$ where \bar{x} is the complex conjugate of x in \mathbf{C} . The orientation of $D \times I$ is preserved under h_2 . The third is defined by $h_3(x, t) = (\bar{x}, t)$. The orientation of $D \times I$ is reversed under h_3 .

Let ρ be the rotation of D determined by ν/μ and let ρ' be the rotation determined by $-\nu/\mu$. The two rotations are inverses of each other. Let T and T' be the fibered solid tori constructed from ρ and ρ' respectively, oriented consistently with $D \times I$. We will show that h_1 and h_2 induce fiber preserving self homeomorphisms of T and h_3 induces a fiber preserving homeomorphism from T to T' . That h_1 induces the identity on T is trivial. A typical two element set $\{(x, 0), (\rho(x), 1)\}$ in $D \times I$ that maps to one point in T is taken by h_2 to $\{(\bar{x}, 1), (\overline{\rho(x)}, 0)\}$. Since

$$\overline{\rho(x)} = \bar{x}e^{-2\pi i(\nu/\mu)} = \rho'(\bar{x}),$$

the pair $\{(x, 0), (\rho(x), 1)\}$ is taken to $\{(\rho'(\bar{x}), 0), (\bar{x}, 1)\} = \{(\rho'(\bar{x}), 0), (\rho\rho'(\bar{x}), 1)\}$ and identified pairs are carried to identified pairs. Since the fibers $\{x\} \times I$ of $D \times I$ are preserved under h_2 , we have an orientation preserving, fiber preserving self homeomorphism of T . Under h_3 , the two element set $\{(x, 0), (\rho(x), 1)\}$ maps to $\{(\bar{x}, 0), (\overline{\rho(x)}, 1)\} = \{(\bar{x}, 0), (\rho'(\bar{x}), 1)\}$ and pairs identified in T are carried to pairs that are identified in T' . This gives an orientation reversing, fiber

preserving homeomorphism from T to T' . We get a self homeomorphism of T from h_3 if $T = T'$. This happens when $\rho = \rho'$ which is true when $\nu/\mu = 1/2$.

The action on $H_1(\partial T)$ induced by h_1 is the identity. The action of h_2 clearly reverses m . It also reverses l since

$$h_2(L(t)) = (e^{-2\pi i(\nu t/\mu)}, 1-t) = (e^{2\pi i(\nu(1-t)/\mu)}, 1-t).$$

This gives two of the automorphisms of Lemma 1.3.2. Two more are realized when $\nu/\mu = 1/2$. We have

$$h_3(L(t)) = (e^{-2\pi i(t/2)}, t).$$

Recalling that $L(t)$ is to be oriented from 1 to 0, we have that $h_3(L(t)) - L(t)$ is homologous to one full circuit around ∂D in a counterclockwise direction. Since h_3 reverses the direction of m , we have that h_3 realizes one of the remaining automorphisms of Lemma 1.3.2 and a fourth is realized by h_2h_3 . It is trivial that the remaining two are realized by h_3 and h_2h_3 on the ordinary fibered solid torus.

We are now in a position to prove:

THEOREM 1.3.3. *Let T and T' be fibered solid tori determined by ν/μ and ν'/μ' respectively. Then there is a fiber preserving homeomorphism from T to T' if and only if $\nu'/\mu' \equiv \pm\nu/\mu \pmod{1}$. The homeomorphism can be taken to be orientation preserving if and only if $\nu'/\mu' \equiv \nu/\mu \pmod{1}$, and can be taken to be orientation reversing if and only if $\nu'/\mu' \equiv -\nu/\mu \pmod{1}$.*

PROOF The if direction follows from the fact that the defining rotations for T and T' are determined by the fractions $\nu/\mu \pmod{1}$ and $\nu'/\mu' \pmod{1}$ and from the homeomorphisms h_2 and h_3 that we have described above. To prove the only if direction, let h be the promised fiber preserving homeomorphism. There is a (non-fiber preserving) homeomorphism g from T to T' taking meridian to

meridian and longitude to longitude (as chosen above). This preserves orientation by definition. Thus $g^{-1}h$ preserves or reverses orientation as h does. A fiber H of T in ∂T generates the maximal cyclic subgroup $[\nu/\mu]$ in $H_1(\partial T)$. Since $g^{-1}h(H)$ generates the maximal cyclic subgroup $[\nu'/\mu']$ and $g^{-1}h$ preserves the maximal cyclic subgroup generated by the meridian, the only if direction follows from Lemma 1.3.1. ■

Some consequences are as follows. Canonical numerical invariants of an oriented fibered solid torus can be taken to be pairs of integers ν and μ with $(\nu, \mu) = 1$, $\mu > 0$, and $0 \leq \nu < \mu$. (Equivalently, a single numerical invariant can be taken to be a fraction ν/μ with $0 \leq \nu/\mu < 1$. If the fraction is in reduced terms with positive denominator, then the numerator and denominator give the pair of integers just mentioned.) Canonical numerical invariants of unoriented fibered solid tori can be taken to be pairs of integers ν and μ with $(\nu, \mu) = 1$, $\mu > 0$, and $0 \leq \nu \leq \mu/2$. In the unoriented case, $\nu = \mu/2$ implies that ν is a divisor of ν and μ so $\nu = 1$ and $\mu = 2$ and we have $0 \leq \nu < \mu/2$ whenever $\mu > 2$. (A single numerical invariant can be taken to be a fraction ν/μ in the interval $[0, 1/2]$.) The numerical invariants for the (unique) ordinary fibered solid torus are $\nu = 0$ and $\mu = 1$. The only fibered solid tori that admit an orientation reversing, fiber preserving self homeomorphism are the ones determined by $0/1$ (the ordinary fibered solid torus) and $1/2$.

The unique behavior demonstrated by the non-ordinary fibered solid torus determined by $1/2$ and the special case in Lemma 1.3.2 associated with $[1/2]$ will persist throughout these notes and be the cause of various special cases.

1.3.4. Homeomorphisms of fibered tori. We will often have occasion to attach a fibered solid torus to a Seifert fibered space by a homeomorphism defined on the boundary of the fibered solid torus. Here we study what is needed

to specify the attaching map, and to what extent the resulting quotient space is determined.

We need the notion of a fiber preserving isotopy. Let X and Y be spaces fibered by circles. A map $\Phi : X \times I \rightarrow Y$ is a "fiber preserving isotopy" if each Φ_t defined by $\Phi_t(x) = \Phi(x, t)$ is a fiber preserving homeomorphism from X to Y . For a given fiber H of X , this requires that for each t the image of H is a fiber of Y , but it does not require that the image of H be the same fiber of Y for all t . Two maps that are connected by a fiber preserving isotopy will be said to be "fiber isotopic".

The first result applies to the uniqueness of the quotient space. It says that the fiber structure of a fibered solid torus is determined by two curves, a fiber and a meridian.

THEOREM 1.3.4. *Let T and T' be fibered solid tori. Let $h : \partial T \rightarrow \partial T'$ be a fiber preserving homeomorphism. Then h extends to a fiber preserving homeomorphism from T to T' if and only if h takes a meridian of T to a curve homologous to a meridian of T' or its reverse.*

PROOF If h extends to any homeomorphism from T to T' , then h takes a meridian of T to a meridian of T' up to reversal. We consider the other direction. If T'' is another fibered solid torus, and if g is a fiber preserving homeomorphism from $\partial T'$ to $\partial T''$ that extends to a fiber preserving homeomorphism from T' to T'' , then h extends to a fiber preserving homeomorphism from T to T' if and only if gh extends to a fiber preserving homeomorphism from T to T'' . We will use this to replace h by another homeomorphism that is more convenient. Let meridians and longitudes be chosen on T and T' so that when these are used as generators for $H_1(\partial T)$ and $H_1(\partial T')$, then the fibers generate maximal cyclic subgroups $[\nu/\mu]$ and $[\nu'/\mu']$ respectively where each fraction is in the interval

$[0, 1)$, is in reduced terms, and has positive denominator. The isomorphism induced by h on H_1 must be given by one of the matrices of Lemma 1.3.2. As shown in the previous section, there is a fiber preserving homeomorphism from T' to some T'' that also realizes this matrix. (We can chose $T'' = T'$ if the determinant of the matrix is $+1$.) Let g be the restriction of this homeomorphism to $\partial T'$. The action of gh on H_1 is given by the identity matrix since each of the matrices of Lemma 1.3.2 has order two.

We can now assume that the action of h on H_1 is given by the identity matrix and concentrate once again on T and T' . We know that T and T' are quotients of $D \times I$ using a rotation determined by ν/μ for both quotients. Thus we can treat T and T' as the same space and we are working with a fiber preserving self homeomorphism h of ∂T that induces the identity on $H_1(\partial T)$.

If h is the identity map, then h extends as desired. If h is fiber isotopic to the identity map, then a homeomorphism can be constructed that reflects the isotopy on a neighborhood of the boundary and a restriction of the identity on the rest. The result now follows from the next lemma. ■

We need the notion of a crossing curve. Let F be a fibered torus. That is, a torus represented as $S^1 \times S^1$ fibered by the circles $\{x\} \times S^1$. A "crossing curve" for F is a simple closed curve on F that intersects each fiber in exactly one point. A crossing curve is necessarily transverse to all the fibers. If Q is a crossing curve for F , then Q and any one fiber represent generators for $H_1(F)$. All other crossing curves for F represent some $\pm Q + bH$, $b \in \mathbf{Z}$ and each such homology class is represented by some crossing curve.

LEMMA 1.3.5. *Let F be a fibered torus. Let h be a fiber preserving self homeomorphism that induces the identity on $H_1(F)$. Then h is fiber isotopic to the identity.*

PROOF Let Q be a crossing curve for F . We let \tilde{F} be the cover of F determined by the subgroup of $\pi_1(F)$ generated by $[Q]$. Thus \tilde{F} is an open annulus and the curves Q and $h(Q)$ lift to \tilde{F} . The cover \tilde{F} is “fibered” by lines. Since h is fiber preserving, both Q and $h(Q)$ are crossing curves for F . Their lifts are crossing curves for \tilde{F} . A slide along a neighborhood of a fiber in F results in an “equivariant” slide along a neighborhood of a fiber in \tilde{F} . An “equivariant” slide along a neighborhood of a fiber in \tilde{F} induces a slide along a neighborhood of a fiber in F . It is “obvious” that a lift of $h(Q)$ can be taken to a lift of Q by “equivariant” slides of neighborhoods of fibers in \tilde{F} . Thus a fiber isotopy that keeps each fiber invariant setwise carries $h(Q)$ to Q . We assume that h has been altered by this isotopy and now carries Q to Q . A similar exercise is now done to a curve parallel to Q and disjoint from Q . The slides are done so as to keep the function on Q unchanged throughout. We can think of Q as the zero level and the new curve as the half level. This can be repeated for the dyadic rationals and in the limit, we get that h is level preserving. The homeomorphism on Q is of degree one and is isotopic to the identity. This extends to all of F by doing the same on each level. This preserves fibers and deforms h to the identity. ■

Note that if T is a fibered solid torus determined by a fiber and meridian where the meridian happens to be a crossing curve for boundary, then T is an ordinary solid torus.

The next results apply to the existence of the quotient spaces.

LEMMA 1.3.6. *Let F and F' be fibered tori. Let $h : F \rightarrow F'$ be a homeomorphism that takes the homology class of a fiber of F to the homology class of a fiber of F' or its negative. Then h is isotopic to a fiber preserving map.*

PROOF If Q and Q' are crossing curves for F and F' respectively, then $h(Q)$ is homologous to $\pm Q' + bH'$, $b \in \mathbf{Z}$. We can thus replace Q by a new curve

by reversing direction if necessary and adding an integral multiple of H so that after the replacement $h(Q)$ is homologous to Q' . Reversing the direction of H if necessary, we get that the induced isomorphism on H_1 is represented by the identity when the bases used are H and Q for $H_1(T)$ and H' and Q' for $H_1(F')$. Thus h is isotopic to a homeomorphism taking H to H' and Q to Q' . By cutting along $H \cup Q$ and $H' \cup Q'$ we see that the result reduces to the Alexander lemma.

■

LEMMA 1.3.7. *Let F be a fibered torus and let J be a simple closed curve on F that is not null homologous and is not homologous to a fiber. Then there is a fibered solid torus T and a fiber preserving homeomorphism $h : \partial T \rightarrow F$ that takes a meridian to the homology class of J . If F is oriented, then T and h can be chosen so that h is orientation preserving.*

PROOF Choose a simple closed curve K on F so that $\psi(J, K) = 1$. (This is with respect to a given orientation or to an arbitrary one if none is given.) The class of a fiber H is given as $\nu J + \mu K$. Since H is a simple closed curve, $(\nu, \mu) = 1$, and since J is not homologous to H , $\mu \neq 0$. By reversing H , we can assume that $\mu > 0$. Since K can be altered by adding an integral multiple of J to K and still have $\psi(J, K) = 1$, we can get $0 \leq \nu < \mu$. We know how to build a fibered solid torus T with meridian m and longitude l so that a fiber represents $\nu m + \mu l$. A homeomorphism exists from ∂T to F taking m to J and l to K . This preserves the (given or chosen) orientation and takes a fiber of ∂T to a curve homologous to H . The result follows from the previous lemma. ■

1.3.5. Attaching fibered solid tori. Combining the above results, we get:

THEOREM 1.3.8. *Let M be a Seifert fibered space and let F be a torus boundary component of M . Let J be a simple closed curve on F that is not homologous on F to a fiber. Then there is a Seifert fibered space M' that is obtained by sewing a fibered solid torus T to F by a fiber preserving homeomorphism that takes a meridian of T to the homology class of J or its reverse. If M is oriented, we can require that the orientation of M' extends the orientation of M . If M'' is formed in a similar fashion, then there is a homeomorphism from M' to M'' that extends the identity on M .*

PROOF That M' can be formed as required follows from Lemma 1.3.7. To get the orientations to cooperate, the sewing map should take the orientation of ∂T as inherited from T to the reverse of the orientation of F as inherited from M . If M'' is also formed to fit the requirements, then let $h_1 : \partial T_1 \rightarrow F$ and $h_2 : \partial T_2 \rightarrow F$ be the respective sewing maps. The last claim will follow if $h_2^{-1}h_1$ extends to a fiber preserving homeomorphism from T_1 to T_2 . However, this follows from Theorem 1.3.4 since a meridian of T_1 is taken to a meridian of T_2 or its reverse. ■

We do not have to attach a fibered solid torus T to a Seifert fibered space along all of the boundary of T . In the next result, the spaces M_i will turn out to be fiber homeomorphic to M however we are not yet ready to prove this. We can at least prove its independence of the attaching map.

LEMMA 1.3.9. *Let M be a Seifert fibered space and let C be a component of ∂M . Let T_i , $i = 1, 2$, be ordinary solid tori with saturated annuli A_i in ∂T_i . If M_i are formed by attaching T_i to M by fiber preserving homeomorphisms from A_i to saturated annuli A'_i in C , then there is a fiber preserving homeomorphism from M_1 to M_2 that extends the identity on M minus a collar on C .*

PROOF The annuli A'_i are regular neighborhoods of fibers in C . Since the fibers are parallel, there is a fiber preserving isotopy of C carrying A'_1 to A'_2 . This implies the existence of a fiber preserving self homeomorphism of M fixed off a collar on C that carries A'_1 to A'_2 . The conclusion now follows in the standard way from the next lemma. ■

LEMMA 1.3.10. *Let T_i , $i = 1, 2$, be ordinary solid tori, and let A_i be saturated annuli in ∂T_i . Then any fiber preserving homeomorphism f from A_1 to A_2 extends to a fiber preserving homeomorphism \bar{f} from T_1 to T_2 .*

PROOF Each T_i is obtained from $D \times I$ by sewing each $(x, 0)$ to $(x, 1)$. A fiber can be used as a longitude and $\partial D \times \{0\}$ can be used as a meridian. The map f may preserve the orientation of the longitude or reverse it. The orbit space is D . The annuli intersect the meridians in a subinterval and the map f determines a map on ∂D that commutes with f and the projections to D . This extends to D and to all of $D \times I$ and thus all of T_1 by crossing with the identity on I if f preserves the orientation of the longitude, or by crossing with the map taking t to $1 - t$ if the longitude is reversed. The map created f' takes A_1 to A_2 and may not agree with f . However f and f' carry each boundary curve of A_1 to the same boundary curve of A_2 and with the same effect on orientation. By the next lemma, f and f' are fiber isotopic and the result follows. ■

LEMMA 1.3.11. *Let A_i , $i = 1, 2$, be fibered annuli and let f and f' be fiber preserving homeomorphisms from A_1 to A_2 . If f and f' carry each boundary curve of A_1 to the same curve of A_2 and with the same effect on orientation, then f and f' are fiber isotopic.*

PROOF The proof uses the same arguments as those used in Lemma 1.3.5 and is simpler. ■

1.4. The orbit space of a Seifert fibered space

Let M be a Seifert fibered space. If we give the set of fibers of M the quotient topology, then we have formed the "*orbit surface*" of M . We have not yet shown that the quotient space is a surface so the terminology is a little premature. The facts will catch up shortly.

1.4.1. The orbit space of a fibered solid torus. Let T be a fibered solid torus defined by ν/μ . Let the set of fibers τ of T have the quotient topology. Regarding T as a quotient of $D \times I$, we identify D with the disk in T that is the image of $D \times \{0\}$. Each fiber of T intersects D at least once, so τ is a quotient of D . If ρ is the rotation of D that is used to define T , then the points of D that lie in the same fiber are the points of an orbit of ρ . Thus τ is the quotient of D under the action of ρ . This quotient is a disk and the quotient map is a branched cover, with branch set the center point of D . The center point of D is carried to an interior point of τ that we think of as the center point of τ . In the complement of the center, the quotient map is a μ -fold covering projection. Thus the projection from T to τ takes points in the boundary of T to the boundary of τ , points in $\overset{\circ}{T}$ to $\overset{\circ}{\tau}$, and the centerline of T to the center point of τ . From this it follows that the orbit surface of a Seifert fibered space is a 2-manifold with boundary and that the projection map carries boundary to boundary and interior to interior. The topological type of the orbit surface is an invariant of the fiber type of the Seifert fibered space.

If T is a fibered solid torus, and τ is its orbit space, then we have realized τ as the result of a composition of two quotient maps. The first map is from T to D . This is not "fiber preserving" unless T is an ordinary solid torus. The second map is the quotient map under the action of ρ .

There is a second way to realize τ as the result of a composition of two quotient maps. In the second way, both maps "preserve fibers." Define $\rho \times 1 : D \times I \rightarrow$

$D \times I$ by $(\rho \times 1)(x, t) = (\rho(x), t)$. This induces a fiber preserving map of order μ on T . The quotient space \overline{T} of this action is naturally fibered as a product $\tau \times S^1$ (an ordinary solid torus). The quotient map q is a branched cover with branch set the centerline of T . Off the branch set, q is a μ -fold covering projection. We now obtain τ by projecting the product $\overline{T} = \tau \times S^1$ onto the first factor.

Let $h_t(\tau)$, $0 \leq t \leq 1$, be an isotopy of τ in that each h_t is an embedding from τ into τ . Further require that each h_t fix the center point of τ . We get an isotopy $h_t \times 1 : \overline{T} \rightarrow \overline{T}$ by setting $(h_t \times 1)(x, s) = (h_t(x), s)$ which fixes the centerline of \overline{T} pointwise. This is a fiber preserving isotopy in that fibers are preserved for each t . For each (x, s) , we have a path $(h_t \times 1)(x, s)$ as t varies from 0 to 1. If x is not the center of τ , then the path avoids the center of $\tau \times \{s\}$. If x is the center of τ , then the path is constant. If (y, s) represents a point in T with $q(y) = x$, then there is a lift $(h_{y,t} \times 1)(y, s)$ to $D \times \{s\}$ starting at (y, s) . This defines an isotopy of T that preserves levels, preserves fibers and fixes the centerline pointwise. By its construction, it is constant on any set $p^{-1}(S)$ where h_t is constant on S and the map $p : T \rightarrow \tau$ is projection.

LEMMA 1.4.1. *Let T be a fibered solid torus and let $p : T \rightarrow \tau$ be the projection of T to its orbit surface. Let τ' be a disk in τ that contains the center point in its interior. Let $T' = p^{-1}(\tau')$. Then there is a fiber preserving homeomorphism from T to T' that is fixed on a neighborhood of the centerline of T . Further, the homeomorphism is isotopic to the identity on T .*

PROOF There is an isotopy connecting the identity of τ to a homeomorphism from τ to τ' . We can require that the isotopy fix pointwise a neighborhood of center point of τ . Lifting this to T gives a fiber preserving isotopy that fixes pointwise a neighborhood of the centerline, and that connects the identity on T to a homeomorphism to T' . ■

1.4.2. Invariance, invariants and transitivity of fibers. We are ready to show the uniqueness of fiber solid torus neighborhoods of fibers.

Let M be a Seifert fibered space and let H be a fiber in the interior of M . Let N and N' be fibered solid tori neighborhoods of H . There is a fibered solid torus neighborhood N'' of H small enough to lie in the interior of $N \cap N'$. If τ is the orbit surface of N , then the image of N'' is a disk τ'' in τ that contains the center point of τ in its interior. By Lemma 1.4.1, we obtain a fiber preserving homeomorphism from N to N'' that is the identity on a neighborhood of the centerline. Similarly, there is such a homeomorphism from N' to N'' . Composing one with the inverse of the other gives a fiber preserving homeomorphism from N to N' preserving H that is the identity on a neighborhood of H . We have:

THEOREM 1.4.2. *Let M be a Seifert fibered space and let H be a fiber in the interior of M . Then any two fibered solid torus neighborhoods of H are connected by a fiber preserving homeomorphism that is the identity on a neighborhood of H . ■*

For fibers in the boundary of a Seifert fibered space, no such theorem is necessary since fibered solid torus neighborhoods of such fibers are ordinary solid tori by definition.

If M is a Seifert fibered space, we will want to gain information about preimages of disks in its orbit surface. The next lemma is a building block.

LEMMA 1.4.3. *Let T be fibered solid torus and let T' be an ordinary solid torus. If T and T' are sewn along saturated annuli in their boundaries by a fiber preserving homeomorphism, then the result is fiber homeomorphic to T .*

PROOF By Lemma 1.3.9 we only have to show this for one particular sewing. Let $p : T \rightarrow \tau$ be the projection of T to its orbit surface. Let E be a disk in

τ that misses the image of the centerpoint and that intersects $\partial\tau$ in an arc. As argued in §1.2.1, $p^{-1}(E)$ is an ordinary solid torus that is sewn to $p^{-1}(\tau - \overset{\circ}{E})$ along a saturated annulus. But $p^{-1}(\tau - \overset{\circ}{E})$ is fiber homeomorphic to T and the result follows. ■

A similar argument gives:

LEMMA 1.4.4. *Let M be Seifert fibered space and let T' be an ordinary solid torus. If M and T' are sewn along saturated annuli in their boundaries by a fiber preserving homeomorphism, then the result is fiber homeomorphic to M . ■*

If M is a Seifert fibered space, and H is a fiber of M , then we say that H is an "ordinary fiber" of M if it has a fibered solid torus neighborhood that is an ordinary solid torus. If not, we say that H is an "exceptional" fiber of M . Since every fiber of a fibered solid torus other than the centerline is ordinary, it follows that the exceptional fibers are isolated in M . We also know that the exceptional fibers are located in $\overset{\circ}{M}$. From Theorem 1.4.2, we know that the numerical invariant ν/μ of a fibered solid torus neighborhood of a fiber H of M is also an invariant of H .

The denominator of ν/μ , if the fraction is in reduced terms, gives the intersection number of a meridian and a fiber in the boundary of a fibered solid torus neighborhood of H . Thus a fiber in the boundary of a fibered solid torus neighborhood represents μ times a generator of π_1 of the neighborhood, while H represents a generator. Thus fibers close to H are " μ times as long" as H . We say that μ is the "index" (or "order") of H .

Let M be a Seifert fibered space, and let G be its orbit surface. A point in G is called "exceptional" if it is the image of an exceptional fiber, and it is called "ordinary" if not. An exceptional point has a neighborhood in which it is the only exceptional point. Thus the exceptional points are isolated in G .

Associated with an exceptional point is the invariant ν/μ of its corresponding fiber. The topological type of G , the number of exceptional points of G and the collection of invariants (with multiplicities) ν/μ associated with the exceptional points are all invariants of the fiber type of M . These are not enough however to classify M , and we will have to further investigate the structure of M .

We return to the task of extracting information from the orbit surface.

THEOREM 1.4.5. *Let M be a Seifert fibered space and let G be the orbit surface of M . Let H be a fiber in M , let h be its image in G , and let E be a disk in G containing h in its interior. If E contains no exceptional point except possibly for h , then the preimage of E in M is a fibered solid torus neighborhood of H .*

PROOF By Theorem 1.4.2, the theorem is true if there is a fibered solid torus neighborhood of H which contains the preimage of E . If not, then E can be triangulated so that h is in the interior of some 2-cell E' of the triangulation, and so that each 2-cell of the triangulation is in the orbit disk of some fibered solid torus in M . Then the preimage of each 2-cell other than E' is an ordinary solid torus, and the preimage of E' is a fibered solid torus neighborhood of H . If there is an ordering of the 2-cells of the triangulation with E' as the first so that each 2-cell intersects the union of its predecessors in an arc, then the result follows from repetitions of Lemma 1.3.9. However, such triangulations of arbitrarily small mesh are easy to find. ■

The next result shows that ordinary fibers in a Seifert fibered space are all pretty much the same. We say that an isotopy of a space X carries one subset S_0 to another S_1 if there is an isotopy $\Phi : X \times I \rightarrow X$ for which Φ_0 is the identity on X and for which $\Phi_1(S_0) = S_1$. Note that if M is a Seifert fibered space, $p : M \rightarrow G$ is the projection to the orbit surface and $\Phi : M \times I \rightarrow M$ is a fiber preserving isotopy of M , then $p\Phi : G \times I \rightarrow G$ defined by $p\Phi(x, t) = p\Phi(p^{-1}(x), t)$ is a well defined isotopy of G .

THEOREM 1.4.6. *Let M be a connected Seifert fibered space and let H and H' be ordinary fibers in M . Then there is a fiber preserving isotopy Φ of M that carries H to H' . If $p : M \rightarrow G$ is the projection to the orbit surface and α is a path from $p(H)$ to $p(H')$ that avoids all exceptional points, then $p\Phi$ can be required to carry $p(H)$ to $p(H')$ along α . If T and T' are fibered solid torus neighborhoods of H and H' respectively, then Φ can be required to carry T to T' .*

PROOF An isotopy of the surface carrying $p(T)$ to $p(T')$ while carrying $p(H)$ to $p(H')$ along α can be constructed in a finite number of stages so that each stage move points only in a disk that contains no exceptional points. (For example, the first and last stages can be based on isotopies that squeeze T and T' to very thin fibered solid torus neighborhoods of H and H' respectively and the other stages can be restricted to solid tori over a chain of small disks that cover the path α .) The preimage of such a disk is an ordinary solid torus and the result follows by lifting these isotopies to M by the techniques of §1.4.1. ■

COROLLARY 1.4.6.1. *Let M be a connected Seifert fibered space and let H and H' be ordinary fibers in M with fibered solid torus neighborhoods T and T' respectively. Then there is a fiber preserving homeomorphism from $M - \mathring{T}$ to $M - \mathring{T}'$. If M is orientable, then the homeomorphism can be chosen to be orientation preserving.* ■

Uniqueness of fibered solid torus neighborhoods can be strengthened.

THEOREM 1.4.7. *Let M be a Seifert fibered space and let H be a fiber of M . If N and N' are fibered solid torus neighborhoods of H , then there is a fiber preserving isotopy of M fixed on a neighborhood of H that carries N to N' .*

PROOF This follows from a similar statement that assumes that N' is contained in N so we make that assumption. The image of N in the orbit surface

G of M is a disk E containing no exceptional point except possibly the image h of H . There is a disk neighborhood E'' of E with the same property. The preimage of E'' is another fibered solid torus neighborhood of H . An isotopy of G fixed on a neighborhood of the image of H and off E'' carries E to E' the image of N' . A lift of this isotopy is the desired isotopy. ■

COROLLARY 1.4.7.1. *Let M be a Seifert fibered space and let H be a fiber of M . If N and N' are fibered solid torus neighborhoods of H , then $M - \mathring{N}$ and $M - \mathring{N}'$ are fiber homeomorphic.* ■

1.4.3. Properties of the orbit surface. We give some remarks connecting the topology of M and its orbit surface G . If M is connected, we know that G is connected. The converse is also true. Paths in G obviously lift locally to M , so the quotient map has (non-unique) path lifting. Now if G is connected, then any two fibers of M are connected by a path, and M is connected. If M is compact, then so is G . If G is compact, then the images of a finite number of neighborhoods of fibers of M cover G . Thus M is the union of a finite number of fibered solid tori and is compact. If M is compact, then it has only a finite number of exceptional fibers and G has only a finite number of exceptional points. From previous remarks, we have that G has no boundary if and only if M has no boundary. Combining all three observations, M is a closed, connected 3-manifold if and only if G is a closed, connected surface. We add an observation about π_1 .

LEMMA 1.4.8. *Let M be a connected Seifert fibered space, let G be the orbit surface of M , and let p be the projection map between them. Then p induces a surjection on π_1 .*

PROOF Paths in G lift to paths in M , so loops in G lift to paths in M that start and end in the same fiber. Such a path can be completed in the fiber to cover the original loop in G . ■

A consequence of this lemma is that any Seifert fibering of S^3 must have S^2 as the orbit space. A Seifert fibering of a closed, connected 3-manifold with finite fundamental group must have S^2 or P^2 as the orbit space.

1.5. Examples

We first consider lens spaces. In dimension 3, a lens space can be viewed as a union of two solid tori sewn together along their boundaries by a homeomorphism. We first look at lens spaces without considering fiber structures.

Let T_1 and T_2 be the two solid tori that make up a lens space L , and think of the sewing of their boundaries as determined by a homeomorphism h from ∂T_1 to ∂T_2 . Once generators for the homologies of the boundaries are chosen, h can be given as a 2×2 integer matrix of determinant ± 1 since the homeomorphism type of L is determined if h is determined up to isotopy. We chose the generators to be meridian-longitude pairs (m_i, l_i) for each torus T_i . The matrix $A = \begin{pmatrix} q & r \\ p & s \end{pmatrix}$ of h states that the image of m_1 is $qm_2 + pl_2$ and the image of l_1 is $rm_2 + sl_2$. The choice of letters is made to agree with standard conventions.

If h_1 is the restriction to ∂T_1 of a self homeomorphism of T_1 and h_2 is a restriction to ∂T_2 of a self homeomorphism of T_2 , then $h_2 h_1$ determines the same space L as h . Self homeomorphisms of T_i exist that reverse the meridians and fix the longitudes, reverse the longitudes and fix the meridians, or reverse both. Thus we may multiply any row or column of A by -1 . This allows us to negate any two entries of A simultaneously. (A diagonal is negated by negating one row and one column.) We may thus assume that $p \geq 0$ and $|A| = 1$. There are self homeomorphisms of T_1 (twists) that fix the meridian and add integer

multiples of the meridian to the longitude. A matrix for this is $\begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}$ which combined with A gives

$$\begin{pmatrix} q & r \\ p & s \end{pmatrix} \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} q & bq+r \\ p & bp+s \end{pmatrix}.$$

However, the right column of the result gives all possible right columns which combine with $\begin{pmatrix} q \\ p \end{pmatrix}$ to give determinant 1. Thus all possible matrices of determinant 1 with left column $\begin{pmatrix} q \\ p \end{pmatrix}$ determine the same space which we refer to as $L_{p,q}$. Similar self homeomorphisms of T_2 combined with A give

$$\begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} \begin{pmatrix} q & r \\ p & s \end{pmatrix} = \begin{pmatrix} q+bp & r+bs \\ p & s \end{pmatrix}.$$

Thus q need only be specified mod p . Since the main diagonal can be negated without violating any of our conventions, q need only be specified up to sign. Lastly, we can think of the sewing as being accomplished by attaching T_2 to T_1 instead of the other way around. The matrix of this is $A^{-1} = \begin{pmatrix} s & -r \\ -p & q \end{pmatrix}$, which can be replaced by $\begin{pmatrix} s & r \\ p & q \end{pmatrix}$ by negating the secondary diagonal. Thus q can be replaced by s which is determined up to multiples of p by the equivalence $qs \equiv 1 \pmod{p}$. Since s can be negated, it can be specified by $qs \equiv \pm 1 \pmod{p}$.

Since $\begin{pmatrix} q \\ p \end{pmatrix}$ is the image of m_1 which is null-homotopic in T_1 , the curve $qm_2 + pl_2$ in ∂T_2 becomes trivial in $L_{p,q}$. However this curve represents p times a generator of $\pi_1(T_2)$. Adding T_1 to T_2 is accomplished by adding a 2-handle which is a neighborhood of a meridinal disk, and then adding a 3-handle. Thus $\pi_1(L_{p,q}) = \mathbf{Z}_p$ where we use $\mathbf{Z}_0 = \mathbf{Z}$ and $\mathbf{Z}_1 = \{1\}$. Thus p cannot be altered without changing $L_{p,q}$. We have established all of one direction and part of the converse of the following. The rest of the converse can be done by calculations of Whitehead torsion.

THEOREM 1.5.1. *The lens spaces $L_{p,q}$ and $L_{p',q'}$ are homeomorphic if and only if $p' = p$ and $q' = \pm q^{\pm 1} \pmod{p}$. ■*

We now consider fiber structures. We can get a Seifert fibered structure on $L_{p,q}$ by putting fiber structures on T_1 and T_2 that are compatible with the sewing homeomorphism. If T_1 is a fibered solid torus determined by ν/μ , then a fiber on ∂T_1 represents the element $\begin{pmatrix} \nu \\ \mu \end{pmatrix}$, and under the action of A , will represent the element

$$\begin{pmatrix} \nu' \\ \mu' \end{pmatrix} = \begin{pmatrix} q & r \\ p & s \end{pmatrix} \begin{pmatrix} \nu \\ \mu \end{pmatrix} = \begin{pmatrix} q\nu + r\mu \\ p\nu + s\mu \end{pmatrix}.$$

This fibers $L_{p,q}$ with (possibly) two exceptional fibers of indices μ and $\mu' = p\nu + s\mu$. Since $\begin{pmatrix} r \\ s \end{pmatrix}$ is determined only up to added multiples of $\begin{pmatrix} q \\ p \end{pmatrix}$, there is considerable freedom in choosing ν/μ and ν'/μ' . We thus get infinitely many examples of Seifert fibered spaces that are homeomorphic but not under fiber preserving homeomorphisms. If μ or μ' is 1, then there will be fewer than two exceptional fibers. If $\mu = 1$, then $\nu = 0$ and $\mu' = s$. We know that the possibilities for s are given by $qs \equiv \pm 1 \pmod p$, so that any lens space can be fibered in infinitely many ways with one exceptional fiber. The lens spaces $L_{p,1}$ can be fibered (among other ways) with no exceptional fibers.

The space S^3 is obtained from the sewing $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$. By our remarks above, S^3 is also obtained from the sewings $\begin{pmatrix} q & sq-1 \\ 1 & s \end{pmatrix}$ for any q and s . Here $\begin{pmatrix} \nu' \\ \mu' \end{pmatrix} = \begin{pmatrix} q\nu + (sq-1)\mu \\ \nu + s\mu \end{pmatrix}$ which can be replaced by $\begin{pmatrix} -\mu \\ \nu + s\mu \end{pmatrix}$ since ν'/μ' is only determined mod 1. Since s is arbitrary, and ν is only required to be relatively prime to μ , it is seen that S^3 can be fibered with two exceptional fibers of any two indices as long as they are relatively prime, with one exceptional fiber of any index, or with no exceptional fibers. We will see later that there are no other possibilities. (Actually, it seems that we won't.)

The space $S^2 \times S^1$ is obtained from the sewing $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ and in fact from any of the sewings $\begin{pmatrix} 1 & r \\ 0 & 1 \end{pmatrix}$. This gives $\begin{pmatrix} \nu' \\ \mu' \end{pmatrix} = \begin{pmatrix} \nu + r\mu \\ \mu \end{pmatrix}$ which can be replaced by

$\begin{pmatrix} \nu \\ \mu \end{pmatrix}$. Thus $S^2 \times S^1$ can be fibered with two exceptional fibers having identical invariants or with no exceptional fibers.

Note that all lens spaces have an orbit surface that is a union of two disks sewn along the boundaries. Thus the orbit surface is S^2 . If G is any closed surface, then $G \times S^1$ is a Seifert fibered space with no exceptional fibers and with orbit surface G . Note that $G \times S^1$ is non-orientable if G is non-orientable. A Seifert fibered space need not be orientable even though each fiber has an orientable neighborhood and is thus an orientation preserving curve. The Seifert fibered space might be orientable even if the orbit surface is not. If G is a non-orientable surface, then there is an orientable circle bundle over G . It can be realized as the double of the orientable interval bundle over G . It contains a copy of the orientable double cover of G (the boundary of the orientable interval bundle over G) as a separating surface. If G is P^2 , then this separating surface is S^2 and the complementary domains are orientable line bundles over P^2 which are copies of P^3 minus a point. Thus the orientable circle bundle over P^2 is the connected sum of two copies of P^3 . We will see that this is the only Seifert fibered space that is not prime. The only other Seifert fibered space that is not irreducible is $S^2 \times S^1$.

If M is the orientable circle bundle over the Klein bottle, then an alternate fibering exists for M . The Klein bottle is realized as a quotient of $S^1 \times I$ in which reflection of S^1 across an axis is used to sew the 0 level to the 1 level. The orientable circle bundle is realized as a quotient of $S^1 \times S^1 \times I$ in which reflection of each S^1 factor across an axis is used to sew the 0 level to the 1 level. Reflection one S^1 reverses orientation, while reflecting two preserves orientation. If we think of S^1 as the unit circle in \mathbf{C} , then reflection can be realized by complex conjugation. Thus each factor has 1 and -1 as two fixed points, and the action on $S^1 \times S^1$ has four fixed points. All other points have

order two and M is seen to have a Seifert fibration with four exceptional fibers each with index 2. The orbit surface has $S^1 \times S^1$ as a 2-fold branched cover with four branch points. If the orbit surface is decomposed as a cell complex using the four exceptional points as vertices and using e edges and f faces, then the branched cover will have four vertices, $2e$ edges and $2f$ faces. Since the Euler characteristic of a torus is 0, we have $4 - 2e + 2f = 0$ and $-e + f = -2$. Thus the orbit surface has Euler characteristic $4 - e + f = 2$ and is S^2 . Thus M fibers over S^2 with four exceptional fibers of index 2, and it also fibers over the Klein bottle with no exceptional fibers.

1.6. Fiber structures of compact Seifert fibered spaces

We now restrict attention to Seifert fibered spaces that are connected, compact manifolds. Such spaces have a finite number of exceptional fibers, and a finite number of boundary components, each of which is a compact fibered surface and is thus a torus or a Klein bottle. The orbit surfaces of such spaces are connected, compact surfaces.

Recall that a compact, connected surface is bounded by a finite number (possibly zero) of circles. Such a surface is associated with a unique closed surface that is obtained by sewing disks to each boundary component. Two compact, connected surfaces are said to be "*of the same genus*" if they yield homeomorphic closed surface when disks are added to all boundary components. If the associated closed surface is orientable, then the genus is usually identified by the number of handles $[1 - (\chi/2)]$ where χ is the Euler characteristic of the closed surface. If the closed surface is non-orientable, then the genus is usually identified with the number of projective plane summands, called the number of crosscaps (which are actually Möbius bands), in the surface $[2 - \chi]$. A compact, connected surface is characterized by its orientability, its genus (expressed as

either a number of handles or as a number of crosscaps), and the number of its boundary components.

Let M be a connected, compact Seifert fibered space. There are only a finite number of exceptional fibers in M and these are in \mathring{M} . Let H be an exceptional fiber in M , let N be a fibered solid torus neighborhood of H , and let J be a meridian of N . If we remove the interior of N , then a connected, compact Seifert fibered space M_0 with one more torus boundary component than M results. We can create another Seifert fibered space M' by sewing in another fibered solid torus N' using a fiber preserving homeomorphism from $\partial N'$ to the new torus boundary component C of M_0 . The space M' is completely determined if a curve on C is picked out to be a meridian of N' . We are only required to choose the curve to be non trivial in the homology of C and not homologous in C to a fiber. If the curve is chosen to be a crossing curve for C , then N' will be an ordinary solid torus, and M' will have one fewer exceptional fiber than M and the same number of boundary components. (Obviously, a Seifert fibered space with no exceptional fibers can be obtained if this step is repeated a finite number of times.) We can recover M from M' by removing the interior of N' and replacing N . All that we have to do to correctly replace N is remember the location of J . If we do not remember where J was, then we need a technique to recover it.

This establishes an outline. We will build Seifert fibered spaces by adding exceptional fibers one at a time, starting with a space with no exceptional fibers. We will do this by removing the interiors of ordinary solid tori, and then replacing them by fibered solid tori. We will need to study the process of removing and replacing fibered solid tori, we will need to study Seifert fibered spaces having no exceptional fibers, and we will need to establish a mechanism for locating meridians of removed fibered solid tori.

1.6.1. Drilling and filling. Let M be a Seifert fibered space, let H be a fiber in the interior of M , and let N be a fibered solid torus neighborhood of H . We know that the fiber type of $M - \mathring{N}$ depends only on H . We say that $M - \mathring{N}$ is obtained from M by "drilling out" the fiber H . Note that when M_0 is obtained from M by drilling out an ordinary fiber, then the fiber type of M_0 is independent of the specific fiber.

Let M be a Seifert fibered space, let C be a torus boundary component of M , and let J be a simple closed curve in C that is not null homologous and is not homologous in C to a fiber. We know that a fibered solid torus can be sewn to M along C so that J becomes a meridian for the fibered solid torus and we know that the fiber type of the result M' depends only on J . We say that M' is obtained from M by "filling" the torus boundary component C .

Note that drilling removes a disk from the orbit surface of a Seifert fibered space and filling adds a disk. Thus if a Seifert fibered space is altered by an equal numbers of drillings and fillings, then the orbit surface of the result is homeomorphic to the orbit surface of the original. If an unequal number of drillings and fillings is done, then the genus of the orbit surface remains the same. Also, drilling and filling will not change the number of boundary components that are Klein bottles. There is more that is invariant. To see this, we need to define another structure.

1.6.2. Classes preserved by drilling and filling. The next lemma is the basis of this section. Recall that if Φ is a fiber preserving isotopy of a Seifert fibered space M and $p : M \rightarrow G$ is the projection to the orbit surface, then $p\Phi$ is the induced isotopy of G . The action of Φ on a fiber H of M gives rise to a path $(p\Phi_t)|_{p(H)}$ in G .

LEMMA 1.6.1. *Let M be a Seifert fibered space, let H be an ordinary fiber of M in \mathring{M} , let $p : M \rightarrow G$ be the projection to the orbit surface, and let Φ be a fiber*

preserving isotopy of M that carries H to a fiber H' in M . Then the unpointed homotopy class of the map $\Phi_1|_H : H \rightarrow H'$ is determined by the homotopy class of the path $(p\Phi_t)|_{p(H)}$ rel its endpoints.

PROOF Since the maps on the fibers are homeomorphisms, there are only two unpointed homotopy classes and these are determined by the effect of the maps on the orientations of the fibers. If counterexamples exist (two isotopies that induce the same path starting at some fiber H , but whose 1 levels give maps on H that are not homotopic), then following one isotopy with the reverse (in t) of the second isotopy would give an example of a fiber preserving isotopy that carries H to itself in an orientation reversing way along a loop in G that is null homotopic in G . In particular this would demonstrate that the element of $\pi_1(M)$ represented by the fiber would be conjugate to its inverse. This cannot happen if M is a fibered solid torus since π_1 of a fibered solid torus is \mathbf{Z} in which a fiber is a non-trivial element and is not conjugate to its inverse. The easiest way to enlarge this observation to an arbitrary M without heavy details is to view an isotopy as a path in a function space.

We are only concerned with the restriction of the isotopy to the fiber H . We thus want to look at the space A of all embeddings of H into fibers of M . Since we are only concerned with the homotopy class of a given embedding, in which any element in a class can be obtained from any other by composing on the right with an orientation preserving self homeomorphism of H , we can let B be the space of all orientation preserving self homeomorphisms of H and let it act on A on the right. The isotopies that we are concerned with induce paths in the space A/B . The space A/B is a double cover of G since it has two points for each fiber in M . We want to know if paths in G lift uniquely to A/B . However, our observation that this is the case if M is a fibered solid torus says that lifting is locally unique, therefore it is globally unique. ■

Using the lemma above, we can construct a well defined homomorphism from $\pi_1(G, x)$ to \mathbf{Z}_2 (regarded as $\{-1, 1\}$ under multiplication) where x is an ordinary point of \mathring{G} . If α is a loop (which can be taken to avoid exceptional points), then we take α to $+1$ if an isotopy carrying H , the fiber above x , to itself around α maps H to itself in an orientation preserving way, and to -1 if it is carried in an orientation reversing way. The lemma shows that this is well defined, and it is clearly a homomorphism. Its kernel is the image of π_1 of the space A/B of the proof of the lemma since a loop lifts to a loop in A/B if and only if the loop is taken to $+1$. We denote this homomorphism by ϕ and call it the "*classifying homomorphism*" of M .

We say that two such homomorphisms $\phi_1 : \pi_1(G_1) \rightarrow \mathbf{Z}_2$ and $\phi_2 : \pi_1(G_2) \rightarrow \mathbf{Z}_2$ are "*equivalent*" if there is a homeomorphism $h : G_1 \rightarrow G_2$ so that $h_\# : \pi_1(G_1) \rightarrow \pi_1(G_2)$ gives $\phi_1 = \phi_2 h_\#$. Fiber homeomorphic Seifert fibered spaces have equivalent classifying homomorphisms.

The classifying homomorphism distinguishes the torus boundary components from the Klein bottle boundary components. If C is a compact boundary component of a Seifert fibered space M , then its image on the orbit surface is a boundary circle. If a loop is conjugate to this circle then its image under the classifying map is $+1$ if C is a torus, and -1 if C is a Klein bottle.

LEMMA 1.6.2. *Let M be a Seifert fibered space and let M' be obtained from M by an equal number of drillings and fillings. Then M and M' have equivalent classifying homomorphisms.*

PROOF If the number of drillings and fillings is zero, then there is nothing to prove. Otherwise there is a consecutive pair of operations one of which is a drilling and the other of which is a filling. We first consider the case where the filling comes first.

There are three spaces involved. Space 1 exists just before the filling, space 2 after the filling and space 3 after the drilling. Since exceptional fibers may be involved, there may not be an easy way to relate the three Seifert fibered spaces, but we can relate the three orbit surfaces. Let these be denoted G_1 , G_2 and G_3 in order. Let E_1 be the disk in G_2 that is the disk added during the filling, and let E_2 be the disk that is removed during the drilling. There is an isotopy of G_2 carrying E_1 to E_2 so we get a homeomorphism f from $G_1 = G_2 - \mathring{E}_1$ to $G_3 = G_2 - \mathring{E}_2$ that extends to a homeomorphism of G_2 that is isotopic to the identity.

If l is a loop in G_1 , then fl is a loop in G_3 and we wish to show that the classifying homomorphisms on G_1 and G_3 take the same values on these two loops. But the classifying homomorphism on G_2 does give these two loops the same value. The value of l on G_1 agrees with that on G_2 since the only change is filling a hole whose boundary had value $+1$. Similarly the value of fl on G_2 agrees with that on G_3 and the classifying homomorphisms on G_1 and G_3 are equivalent.

It the drilling comes first, then the analysis is similar except that now G_2 has two boundary components J and K (perhaps the same), one filled with a disk to create G_1 and the other filled with a disk to create G_3 . The classifying homomorphisms must take the value $+1$ on both J and K since the boundary components over J and K are involved in drilling and filling and must be tori. We can form G'_2 by attaching disks to both J and K (if $J = K$ only one disk is attached) and extending the classifying homomorphism. There is an ambient isotopy of G'_2 that carries one attached disk to the other and the remaining details are left to the reader.

The general induction step now involves surfaces G'_1 , G_1 , G_2 and G'_2 with classifying homomorphisms where the homomorphisms on G_1 and G_2 are known

to be equivalent and each G'_i is obtained from G_i by removing a disk or each G'_i is obtained from G_i by adding a disk. The details are left to the reader. ■

There are restrictions on the values of a classifying homomorphism on boundary components. Compact surfaces are built from punctured disks, once punctured tori and Möbius bands. The boundary of a once punctured torus is a commutator and the boundary of a Möbius band is a square. Thus these curves must be taken to $+1$ by a classifying homomorphism. The curves on the boundary of a punctured disk are related by having their product equal 1. Thus it cannot be that an odd number of them is taken to -1 by a classifying homomorphism. A compact surface can be regarded as a punctured disk which has some of its boundary components sewn to once punctured tori and Möbius bands. The curves that are used for attaching have value $+1$ under a classifying homomorphism and an even number of the remaining curves (possibly zero) have value -1 . Thus the number of Klein bottle boundary components of a compact Seifert fibered space is even.

Let M be a connected, compact Seifert fibered space. We define the "*class*" of M to be the class of all closed Seifert fibered spaces that can be obtained from M by a finite sequence of drillings, fillings and fiber preserving homeomorphisms. We would like to pick out a well defined representative of a class. Since all exceptional fibers can be drilled, we can always get a space with no exceptional fibers. Since all torus boundary components can be filled, we can always get a space with no torus boundary components. Unfortunately, a class can have more than one fiber type of space with no exceptional fibers and no torus boundary components. However, we will see that this is a problem only of closed manifolds. As soon as boundary is introduced, ambiguity goes away. We therefore define (the) "*classifying space*" of a class to be (the) representative with no exceptional fibers and exactly one torus boundary component. We could do without the torus

boundary component if there are Klein bottle boundary components present, but there seems to be no gain by doing that. Uniqueness must be established. We need some techniques.

Let G be a compact, connected surface with non-empty boundary and let ϕ be a homomorphism from $\pi_1(G)$ to \mathbf{Z}_2 . The basepoint is no problem because \mathbf{Z}_2 is abelian and ϕ is impervious to change by conjugation. There are a finite number of pairwise disjoint arcs $\{\alpha_i\}$ in G with boundaries in the boundary of G so that cutting G along these arcs yields a disk E . The surface G is recovered by sewing pairs of arcs together in the boundary of E . These arcs $\{\alpha'_i, \alpha''_i\}$ in ∂E are pairwise disjoint and have two copies α'_i, α''_i corresponding to each α_i . Let M_0 be a Seifert fibered space with no exceptional fibers and with orbit surface E . We know that M_0 must be an ordinary solid torus. We fix one fiber in ∂M_0 as a longitude with a given orientation for future reference. This orients all of the fibers in M_0 by requiring that they all represent the same homology class as the longitude. The preimages of the α'_i and α''_i in M_0 are a set of pairwise disjoint saturated annuli in ∂M_0 . We recover a Seifert fibered space whose orbit surface is G if we sew these annuli together in pairs with some care. With some extra care, we also get a space whose classifying homomorphism is ϕ . The sewings have to be with fiber preserving homeomorphisms. The care needed to recover G is to see that the sewing of the annuli reflects the sewing of the α'_i to the α''_i . This is accomplished by seeing that the right pairs of annuli are sewn together, and seeing that orientations of the I factors in the annuli are handled to reflect the orientations of the sewings of the α'_i to the α''_i . This requires that the boundaries of the annuli be matched correctly. To recover ϕ , we need to handle the orientations of the S^1 factors correctly. This is done by finding a simple closed curve J in G that pierces a given α_i once and noting the value of ϕ on J . If the value is $+1$, then the orientations of the S^1 fibers in the

appropriate annuli are to be sewn so as to match the orientations as inherited from the fixed longitude of M_0 . If the value is -1 , then the orientations of the fibers in one annulus are to be matched to the negatives of the orientations of the fibers in the other. This will create a space with the right value of the classifying homomorphism on J . Since a set of curves chosen one per α_i that pierces α_i once is a set of generators for $\pi_1(G)$, we have reconstructed ϕ .

We call this construction a realization of ϕ along the arcs $\{\alpha_i\}$. Note that the space constructed has no exceptional fibers. Since all realizations start with an ordinary solid torus, the only freedom in constructing the realization is in the sewing maps. However, the pairing of the annuli to be sewn is determined by G and the set $\{\alpha_i\}$, the treatment of the orientations of the I factors is determined, and the treatment of the S^1 factors is determined. By Lemma 1.3.11, the sewings used in one realization are fiber isotopic to the sewings used any other realization. Thus any two realizations along $\{\alpha_i\}$ are fiber homeomorphic. It is also clear that every compact, connected Seifert fibered space M with non-empty boundary and an absence of exceptional fibers is a realization because it is obtained from an ordinary solid torus by sewings of certain annuli that are obtained as saturated sets over an appropriate set of cutting arcs of the orbit surface of M .

Let $\{\beta_i\}$ be another set of arcs that cut G to a disk E' . Let M be one realization along $\{\alpha_i\}$. The preimages of the β_i are saturated annuli in M that cut M to a Seifert fibered space M'_0 with no exceptional fibers with orbit surface E' . Thus M'_0 is seen to be an ordinary solid torus and M is seen to be a realization along $\{\beta_i\}$. Thus realizations along $\{\beta_i\}$ are fiber homeomorphic to realizations along $\{\alpha_i\}$.

Now let G and G' be surfaces possessing homomorphisms ϕ and ϕ' respectively, and let $h : G \rightarrow G'$ be a homeomorphism that demonstrates that ϕ and ϕ' are equivalent. Let M realize ϕ along arcs $\{\alpha_i\}$ in G . If $p : M \rightarrow G$ is projection,

then $hp : M \rightarrow G'$ is also a projection and M is seen also to be a realization of ϕ' along $\{h(\alpha_i)\}$ in G' . Thus realizations of ϕ and ϕ' are fiber homeomorphic. We have shown the following.

THEOREM 1.6.3. *(a) Let G be a compact, connected surface with non-empty boundary and let ϕ be a homomorphism from $\pi_1(G)$ to \mathbf{Z}_2 . Then there is a Seifert fibered space with orbit surface G , with no exceptional fibers and with classifying homomorphism ϕ . Any two such are fiber homeomorphic.*

(b) Compact, connected Seifert fibered spaces with no exceptional fibers, with non-empty boundary, and with equivalent classifying homomorphisms are fiber homeomorphic. ■

COROLLARY 1.6.3.1. *Let M be a class of Seifert fibered spaces. Then any two classifying spaces for the class are fiber homeomorphic.*

PROOF Two such spaces are obtained from each other by drillings and fillings. Since the number of torus boundary components of the two spaces is the same, the number of drillings and fillings is equal. Thus the two spaces have equivalent classifying homomorphisms and are fiber homeomorphic. ■

COROLLARY 1.6.3.2. *Two compact, connected Seifert fibered spaces with the same number of torus boundary components are in the same class if and only if they have equivalent classifying homomorphisms.*

PROOF On each space, drillings and fillings replace exceptional fibers by ordinary fibers. For either space, the number of drillings and fillings is the same and the classifying homomorphisms stay in their equivalence classes. If the resulting spaces have boundary, we are done. If not, then one drilling each gives spaces with boundary and the classifying homomorphisms become equivalent if and

only if the originals were equivalent. However, the classifying homomorphisms become equivalent if and only if the spaces become fiber homeomorphic. ■

The spaces of the corollary above need not be fiber equivalent nor even homeomorphic. The examples of fiberings of lens spaces given above are all in the same class. Since the orbit surfaces are all S^2 , which is simply connected, the classifying homomorphisms are all trivial.

The next two corollaries are really corollaries of the realization construction.

COROLLARY 1.6.3.3. *Every compact, connected Seifert fibered space with non-empty boundary and no exceptional fibers admits a fiber preserving self homeomorphism that carries each fiber to itself and reverses the orientations of all the fibers.*

PROOF The realization construction starts with a disk E and an ordinary solid torus that has the structure of $E \times S^1$. We think of S^1 as the unit circle in \mathbf{C} . Annuli $\alpha' \times S^1$ are sewn to annuli $\alpha'' \times S^1$ and the sewings are only specified by orientation requirements on the arcs α' and α'' and on the S^1 factors. We have enough freedom to cover the possibilities if we insist that the sewings in the S^1 direction use the either identity on S^1 or complex conjugation. This commutes with the self map of $E \times S^1$ that carries each (e, x) to (e, \bar{x}) . This induces the desired homeomorphism on the realization. ■

If M is a Seifert fibered space, G its orbit surface and $p : M \rightarrow G$ the projection, then a "section" for G is an embedding $\sigma : G \rightarrow M$ for which $p\sigma$ is the identity on G .

COROLLARY 1.6.3.4. *Every compact, connected Seifert fibered space with non-empty boundary and no exceptional fibers has a section for its orbit surface.*

PROOF As in the proof of the previous corollary, we start with the ordinary solid torus $E \times S^1$ and its section embedding E into $E \times \{1\}$. Since fibers are

sewn using the identity or complex conjugation on the S^1 factor, the sewing of E to recreate G is reproduced by the image of $E \times \{1\}$. ■

We will concentrate on understanding closed, connected Seifert fibered spaces in the next section. The obvious question that remains for this section is how many equivalence classes of homomorphisms from π_1 of a surface to \mathbf{Z}_2 are there. Before we answer that we will show that if we answer this question for closed surfaces, then we will not only have enough information to understand this question for arbitrary surfaces but we will also have enough information for an understanding of compact, connected Seifert fibered spaces modulo an understanding of closed, connected Seifert fibered spaces.

Let G be a compact, connected surface with boundary and let ϕ be a homomorphism from $\pi_1(G)$ to \mathbf{Z}_2 . We know that the number of boundary components on which ϕ takes the value -1 is even. Thus if G has one boundary component, then ϕ takes the value $+1$ on this component and ϕ has a unique extension $\hat{\phi}$ to the surface \hat{G} which is obtained from G by sewing a disk onto the boundary curve. We know that restricting $\hat{\phi}$ to any surface obtained from \hat{G} by removing the interior of a disk results in a homomorphism that is equivalent to ϕ . Thus if ∂G is connected, then the equivalence classes of homomorphisms from $\pi_1(G)$ to \mathbf{Z}_2 are in one-to-one correspondence with the equivalence classes of homomorphisms from $\pi_1(\hat{G})$ to \mathbf{Z}_2 .

If G has several boundary components, then let J be a separating, simple closed curve on G that bounds a disk with holes E containing all the boundary components of G . Let \hat{G} be the result of sewing a disk to $G - \text{Int } E$ along J . A homomorphism from $\pi_1(G)$ to \mathbf{Z}_2 determines a unique $\hat{\phi}$ from $\pi_1(\hat{G})$ to \mathbf{Z}_2 , and a restriction to E . The homomorphism ϕ can be reconstructed from the restrictions of $\hat{\phi}$ to $G - \text{Int } E$ and ϕ to E . As mentioned above, the class of the restriction of $\hat{\phi}$ to $G - \text{Int } E$ is determined by $\hat{\phi}$ and the fact that \hat{G} and

$G - \text{Int } E$ differ by a disk. The restriction of ϕ to E is determined by the number of boundary components of G that ϕ evaluates to -1 . This is because homeomorphisms of E can realize any permutation of the boundary components of E keeping J fixed, and because $\pi_1(E)$ is generated by the boundary components of E with J omitted. Thus the equivalence classes of all homomorphisms from fundamental groups of compact, connected surfaces to \mathbf{Z}_2 are determined by those from closed, connected surfaces, and from those on disks with holes. The classes on closed surfaces will be discussed shortly, and the classes on disks with holes are easily understood.

Now let M be a compact, connected Seifert fibered space with non-empty boundary. Assume that all exceptional fibers have been drilled out and replaced with ordinary solid tori. Let G be the orbit surface, let $p : M \rightarrow G$ be the projection, and let ϕ be the classifying homomorphism. As before, let J be a separating simple closed curve on G that bounds a disk with holes E that contains all the boundary components of G . The saturated torus $T = p^{-1}(J)$ splits M into two Seifert fibered spaces M' with $\partial M' = T$, and M'' with $\partial M'' = T \cup \partial M$. We can sew an ordinary solid torus to M' along T to create a closed, connected Seifert fibered space \hat{M} with no exceptional fibers. The classifying homomorphism of \hat{M} is determined by ϕ . The fiber type of M' is determined by the classifying homomorphism of \hat{M} and the fact that M' is obtained from \hat{M} by drilling out a single fiber.

The space M'' has E as its orbit surface and is determined by the restriction of ϕ to $\pi_1(E)$ which in turn is determined by the number of boundary curves of E that ϕ takes to -1 which equals the number of boundary components of M that are Klein bottles. We recover M as $M' \cup M''$. If spaces fiber homeomorphic to M' and M'' are supplied instead (which we still refer to as M' and M''), then we can try to recover a space fiber homeomorphic to M by sewing M' to M''

by a fiber preserving homeomorphism from one torus boundary component of M'' to the unique boundary component of M' . No matter what fiber preserving homeomorphism is used, the result of the sewing will have orbit surface homeomorphic to G and the homeomorphism can be arranged to demonstrate that the classifying homomorphism is equivalent to ϕ . Thus we always get a space fiber homeomorphic to M .

We now consider the question of determining the equivalence classes of homomorphisms from fundamental groups of closed, connected surfaces to \mathbf{Z}_2 . We will see that for orientable surfaces there are 1 or 2 classes, and for non-orientable surfaces, there are as many as four classes. The fact that there are at least this many classes is shown by constructing the homomorphisms and showing by various arguments that they are not equivalent. The fact that they exhaust all the classes is shown by knowing enough homeomorphisms of closed surfaces to reduce any homomorphism to one of the classes. This last argument is done mostly by pictures.

Let G be a closed, connected, orientable surface with g handles. The fundamental group is generated by $2g$ curves in G with one point in common whose complement in G is an open disk. The only relation that applies to these generators is that a certain product of commutators is the identity. Thus any assignment of ± 1 to each curve results in a valid homomorphism to \mathbf{Z}_2 . There are two obviously different homomorphisms — one ϕ_1 that has image $\{+1\}$ and another ϕ_2 that has image $\{+1, -1\}$. The homomorphism ϕ_1 is trivial and needs no more details. A canonical example for ϕ_2 is to define ϕ_2 to be -1 on each of the $2g$ generators described above. We will argue later that these represent all classes. Note that ϕ_2 does not apply to S^2 .

Let G now be a closed, connected, non-orientable surface with k crosscaps. Not all curves on G are alike. Some preserve orientation and some reverse

orientation. The subgroup O of orientation preserving curves is of index 2 in $\pi_1(G)$. Two homomorphisms that are immediate are the trivial ϕ_1 which takes all of $\pi_1(G)$ to $+1$, and another ϕ_3 that has image all of \mathbf{Z}_2 with kernel O . That is, it takes every orientation reversing curve to -1 . We now find two more examples.

Since \mathbf{Z}_2 is abelian, a homomorphism from $\pi_1(G)$ factors through $H_1(G)$. We will find that there is a unique involution in $H_1(G)$ and will use that to detect differences between homomorphisms. Generators for $\pi_1(G)$ and $H_1(G)$ are a set of k orientation reversing curves in G , one that crosses each crosscap. The only relation in $\pi_1(G)$ is that the product of their squares is the identity. In $H_1(G)$, this becomes the sole relation that the sum of their doubles is zero. If $0 \neq x \in H_1(G)$ has $2x = 0$, then $2x$ is an even multiple of the sum of the the generators, and $x \neq 0$ is an odd multiple of the sum of the the generators. Since twice the sum of the generators is zero, x is reduced to the sum of the generators and is seen to be unique. We now build two more homomorphisms by defining their values on the generators. If there is more than one generator (G is not P^2), then ϕ_4 takes one generator to -1 and all the rest to $+1$. If there are more than 2 generators (G is neither P^2 nor the Klein bottle), then ϕ_5 takes exactly two of the generators to -1 and all the rest to $+1$. Neither ϕ_4 nor ϕ_5 is equal to the trivial ϕ_1 . Neither is equal to ϕ_3 since each is $+1$ on some orientation reversing curve. They are not equivalent since on $H_1(G)$ they disagree on the unique involution x .

If G is a closed, connected surface, we have constructed one homomorphism if G is S^2 , two if G is orientable and not S^2 , two if G is P^2 , three if G is the Klein bottle, and four if G is non-orientable and is neither P^2 nor the Klein bottle.

THEOREM 1.6.4. *If G is a closed, connected surface and $\phi : \pi_1(G) \rightarrow \mathbf{Z}_2$ is a homomorphism, then ϕ is equivalent to one of ϕ_i , $i \in \{1, 2, 3, 4, 5\}$.*

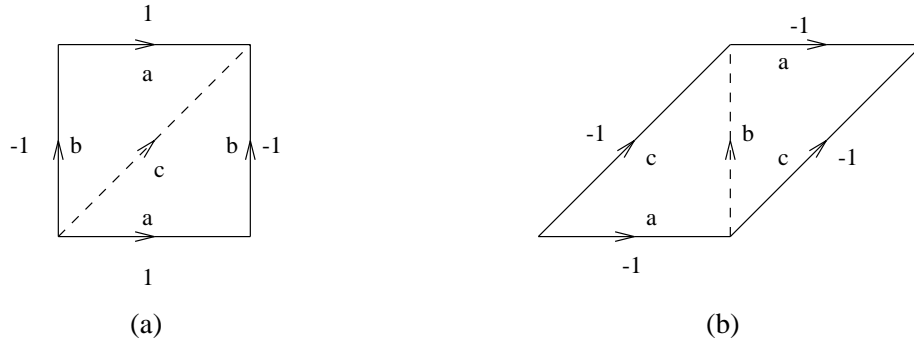


FIGURE 1.2. One orientable handle

PROOF If G is S^2 , P^2 , the Klein bottle, or the connected sum of three copies of P^2 , then the statement follows because there are not enough generators to create more examples. In the other cases, assume that $\pi_1(G)$ has been endowed with the standard generating sets described above. If G is orientable and not S^2 , then what must be shown is that if ϕ takes at least one generator to -1 , then it is equivalent to ϕ_2 . Inductively we will show that if ϕ does not take all generators to -1 , then it is equivalent to a homomorphism that takes more generators to -1 . If G is non-orientable, then what must be shown is that if ϕ takes more than two but not all generators to -1 , then it is equivalent to a homomorphism that takes fewer generators to -1 . These demonstrations will suffice because the homeomorphisms of G are rich enough to take any one set of standard generating curves for $\pi_1(G)$ to any other set of standard generating curves.

If G is orientable and has a handle with one generating curve taken to $+1$ and one to -1 , then we can find two generating curves that are taken to -1 . In Figure 1.2(a) a curve piercing the “side” labeled a is taken to $+1$ and a curve piercing the “side” labeled b is taken to -1 . The handle is homeomorphic to

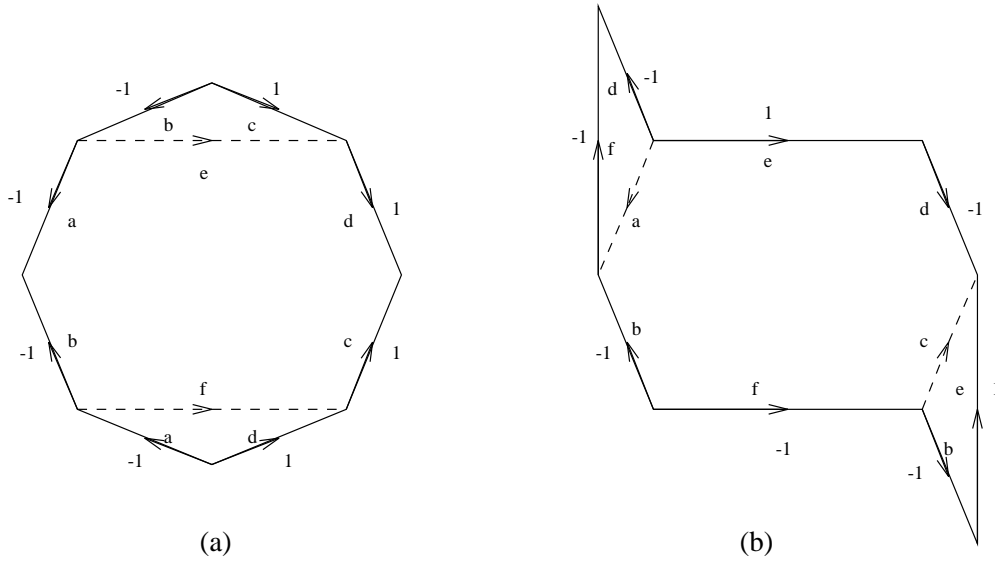


FIGURE 1.3. Two orientable handles

the diagram in Figure 1.2(b) where the generating curves that pierce the two shown sides pass across the line b and both take value -1 .

If G has two handles, one with generators taken to -1 and one with generators taken to $+1$, then we wish to raise the number of generators in the two handles that are taken to -1 to at least three. In Figure 1.3(a), the “sides” (a, b) form a handle whose generators are taken to -1 , and the “sides” (c, d) form a handle whose generators are taken to $+1$. The surface shown is homeomorphic to the diagram in Figure 1.3(b) where (b, f) and (d, e) form handles and the generators piercing b, d and f are taken to -1 , while the generator piercing e is taken to $+1$.

The two changes are enough to complete the induction.

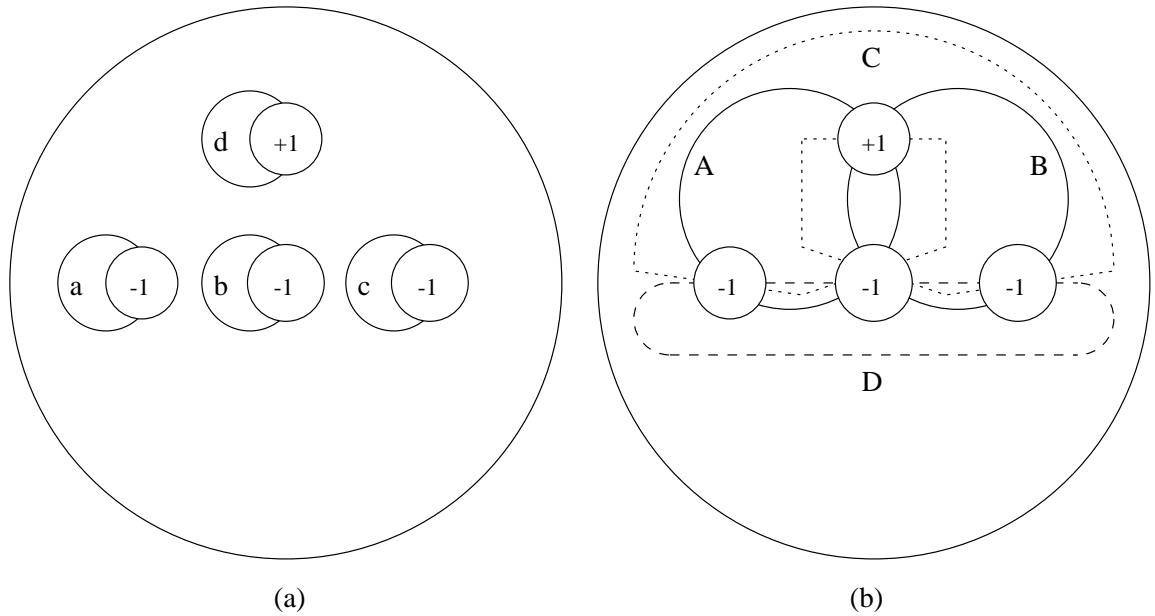


FIGURE 1.4. Four crosscaps

If G is non-orientable, has at least four crosscaps, and has three of the corresponding generators taken to -1 and one taken to $+1$, then we wish to find new generators that have fewer taken to -1 . In Figure 1.4(a), the four small circles represent four crosscaps, and the four labeled curves piercing them are four disjoint, orientation reversing curves that are generators. The values given the various generators are shown in the circles. In Figure 1.4(b), the four labeled curves are also four disjoint, orientation reversing curves (they each pierce an odd number of crosscaps), and there is a homeomorphism taking the diagram of Figure 1.4(a) to Figure 1.4(b) taking the curves (a, b, c, d) to the curves (A, B, C, D) . The values given the new generators can be read from Figure 1.4(b) by counting the number of times they cross the crosscaps with value -1 . Thus A , B and C are taken to $+1$ and D is taken to -1 . That exactly two

fewer generators are taken to -1 is not surprising since the value on the unique involution in $H_1(G)$ has to stay the same. \blacksquare

We consider what this result tells us about classes of Seifert fibered spaces. We look first at the orientability of the spaces in various classes.

Let M be a compact, connected Seifert fibered space, G its orbit surface, $p : M \rightarrow G$ the projection, and $\phi : \pi_1(G) \rightarrow \mathbf{Z}_2$ the classifying homomorphism. Let $\rho : \pi_1(G) \rightarrow \mathbf{Z}_2$ be the homomorphism taking all orientation reversing curves in G to -1 . If G is orientable, then ρ is trivial. Let l be a loop in M . We wish to see if l reverses orientation. Since we can homotop l without changing its behavior on orientation, we ask that l avoid exceptional fibers. We can establish an “orientation frame” along each point of l where two of the directions lie in a meridional disk of a fibered solid torus neighborhood and the third lies in the direction of a fiber. The trace of the two meridional directions can be followed in the orbit surface G as the loop pl runs below l . We get an inversion of orientation of the two meridional directions if pl is an orientation reversing loop in G . That is, if $\rho(pl) = -1$. We get an inversion in the fiber direction if $\phi(pl) = -1$. The orientation of the 3-dimensional space M is reversed if one or the other but not both of these orientations is reversed. Thus l reverses orientation in M if and only if $\rho(pl)\phi(pl) = -1$. This says that M is orientable if and only if the pointwise product $\rho\phi$ takes on only the value $+1$. That is, if and only if $\phi = \rho$. We can now look at cases.

If G is orientable, then ρ is trivial and M is non-orientable if and only if ϕ takes on negative values. Thus if G is S^2 , there is only one class, and it has only orientable spaces. If G is orientable and not S^2 , then there are two possible classes, one with all orientable spaces and the other with all non-orientable spaces. If G is non-orientable, then there are always at least two classes. One, corresponding to trivial ϕ , with non-orientable spaces, and one, with $\phi = \rho$ with

orientable spaces. If G has more than one crosscaps, there is a third class, and if G has more than two crosscaps, there is a fourth class. The spaces in these extra classes are all non-orientable since ϕ and ρ disagree on at least some curves.

Interesting things can happen in the non-orientable spaces. Let M , G , ϕ and ρ be as above with M non-orientable. Let α be a loop in G . Let α be based at an ordinary point x and let H be the fiber in M over x . By Theorem 1.4.6, there is a fiber preserving isotopy Φ of M starting at the identity that drags H along a path of fibers over α . We can require that the isotopy take a fiber solid torus neighborhood T of H back to itself. We know that $\Phi_1 : T \rightarrow T$ reverses orientation if and only if $\rho(\alpha)\phi(\alpha) = -1$. Since M is non-orientable, such an α must exist. We can now prove the following.

LEMMA 1.6.5. *Let M be a non-orientable, Seifert fibered space, let C be a torus boundary component of M , let H be a fiber in C , and let Q be a crossing curve for C . Let n be in \mathbf{Z} . Then there is a fiber preserving, self homeomorphism of M that is fixed on all boundary components of M except C so that the image of Q is homologous to one of $\pm(Q+2nH)$. The homeomorphism can be chosen to be orientation preserving on C , and it can be chosen to be orientation reversing on C . It may not be possible to chose which of $Q + 2nH$ or $-Q + 2nH$ is the image of Q .*

PROOF We can sew an ordinary solid torus T to C and use the isotopy of the discussion above. This will give us a homeomorphism from M to M that has certain effects on C . The ability to make things happen comes from the fact that the way we sew T to C is up to us. We will first get a homeomorphism that reverses the orientation on C , and then we alter it so that it has the same effect on Q but preserves orientation.

Let T be sewn to C so that the meridian of T goes to the crossing curve $Q + nH$. Denote the resulting Seifert fibered space by \hat{M} . We can drag the

centerline of T around an orientation reversing curve of \hat{M} so that T is returned to itself with its orientation reversed. This can be done by an isotopy Φ that moves no points on any of the boundary components of \hat{M} . The map $\Phi_1|_M : M \rightarrow M$ is our first candidate. Since the isotopy is fiber preserving and the meridian of T must be returned to itself, the action of Φ_1 on $C = \partial T$ is mostly determined. We use column vectors to denote $H_1(T)$ and we let (H, Q) be a generating pair for $H_1(T)$. In the simple case where $n = 0$, we are returning Q to $\pm Q$ and H to $\pm H$. Since Φ_1 is orientation reversing on T , the matrix involved must be $\pm \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$.

In the case where $n \neq 0$, then H and $Q + nH$ are each fixed up to sign, and we get the matrix by conjugating by the change of basis matrix that takes H to H and Q to $Q + nH$. The resulting matrix is

$$\pm \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 1 & -n \\ 0 & 1 \end{pmatrix} = \pm \begin{pmatrix} 1 & -2n \\ 0 & -1 \end{pmatrix}.$$

This gives the result with a map $\Phi_1|_M$ that reverses orientation on C . We get a map that preserves the orientation on C by composing $\Phi_1|_M$ with a second orientation reversing map constructed by using the curve $Q + 2nH$ as the image of the meridian of T so that $Q + 2nH$ is preserved up to sign. ■

1.6.3. Closed Seifert fibered spaces without exceptional fibers.

If M is a closed, connected Seifert fibered space with no exceptional fibers, then a single fiber can be drilled out by removing the interior of a fibered solid torus N and we are left with a classifying space M_0 for the class of M . That is, M_0 has no exceptional fibers and has a single torus boundary component. The classifying homomorphism for M_0 is determined by that for M and in turn this completely determines M_0 . Thus M is determined by its classifying homomorphism and the way that N was attached to M_0 . The way that N is attached is important because it turns out that different spaces can be obtained from M_0 by attaching

N in different ways. The lens spaces $L_{p,1}$ that can be realized with no exceptional fibers are examples.

Let N be an ordinary solid torus, and let h be a fiber preserving homeomorphism from ∂N to ∂M_0 . Let M_h be the Seifert fiber space obtained by sewing N to M_0 using h . Let m be a meridian for N and let H be a fiber on ∂N . We can also use H as a longitude. By Theorem 1.3.8, we know that the fiber type of M_h is determined by the homology class (up to sign) of $h(m)$. Since N is an ordinary solid torus, $h(m)$ must be a crossing curve on ∂M_0 . Given one fixed crossing curve Q on ∂M_0 , all other crossing curves Q' have the form $\pm Q + nH$, $n \in \mathbf{Z}$. Since the sign is irrelevant in determining M_h , we have a \mathbf{Z} indexed set of possibilities $Q + nH$ to use for the image of m under h . We will show that for orientable M_0 , these each lead to different fiber types for M_h , and that for non-orientable M_0 these collapse to two types.

We consider the structure of M_0 . By Corollary 1.6.3.4, we know that there is at least one curve Q on ∂M_0 that is a crossing curve and that is the boundary of a section σ of the orbit surface. We will show that if M_0 is orientable, then Q is essentially unique, and if M_0 is non-orientable, then Q is determined only up to an even multiple of a fiber.

Let G_0 be the orbit surface for M_0 and let σ' be another section for G_0 . Let Q' be the curve bounding the image of σ' . We can adjust σ and σ' by sliding along fibers so that Q and Q' are transverse. We are interested in $\psi(Q, Q')$.

Let $\{\alpha_i\}$ be a set of arcs in G_0 that cut G_0 into a disk E . The preimages of these arcs under the projection are saturated annuli A_i in M_0 that cut M_0 into an ordinary solid torus T_0 . After the cut we get two copies α'_i and α''_i for each arc and two copies A'_i and A''_i for each annulus. We can “cut” σ and σ' by restriction into sections for E into T_0 . Let ∂E be given an orientation and let R and R' be the images under the “cut” σ and σ' of ∂E . Let R and R' inherit

their orientations from ∂E . Further adjustment of σ can assure that R and R' are transverse and that they do not intersect at the boundaries of the A'_i and A''_i .

The curves R and R' contain all of Q and Q' broken into various arcs. These arcs are separated by arcs in R and R' that map into the A'_i and A''_i . Thus the intersections of R and R' include all intersections of Q and Q' plus extra intersections that occur inside the A'_i and A''_i . Symbolically we can say

$$(1.1) \quad \psi(R, R') = \psi(Q, Q') + \sum_i \psi(R \cap (A'_i \cup A''_i), R' \cap (A'_i \cup A''_i)).$$

Each intersection of R and R' inside some A'_i has a corresponding intersection in A''_i . Since M_0 is orientable, A'_i is sewn to A''_i by a homeomorphism that reverses the orientations that A'_i and A''_i inherit from T_0 . We need to know how this homeomorphism deals with the orientations of R and R' at these corresponding intersections. Since σ and σ' are sections, each of R and R' intersects an A'_i in a single arc that is transverse to the fibers of A'_i . These arcs are the images of α'_i . These arcs cross A'_i in the same direction in that they originate at the same boundary component of A'_i . The same statements apply to the intersection of R and R' with A''_i . Thus the sewing of A'_i to A''_i sews $R \cap A'_i$ to $R \cap A''_i$ and $R' \cap A'_i$ to $R' \cap A''_i$ in a way that preserves the orientations of both arcs or reverses the orientations of both arcs. Thus intersections of R and R' inside some A'_i are mapped to corresponding intersections in A''_i by a map that behaves the same on the orientations of both R and R' and that reverses the orientations of A'_i and A''_i . Thus the corresponding intersections have opposite sign and cancel in the sum 1.1. This gives $\psi(R, R') = \psi(Q, Q')$. But R and R' are meridians on the solid torus T_0 and have intersection zero. Thus for M_0 orientable, boundaries of sections are homologous up to sign on ∂M_0 .

If M_0 is not orientable, then we do not know how A'_i is sewn to A''_i . Thus intersections of R and R' in A'_i may cancel with intersections in A''_i or double

with them. The only conclusion that we can reach is that $\psi(Q, Q')$ is an even number. Thus for M_0 non-orientable, if Q is a boundary for a section, then any other boundary of a section must be homologous up to sign to $Q + 2nH$, $n \in \mathbf{Z}$. However the converse is also true. Each homology class of the form $Q + 2nH$ is represented by the boundary of a section because for M_0 non-orientable, Lemma 1.6.5 gives a fiber preserving, self homeomorphism of M_0 that takes Q to the class $Q + 2nH$.

The information that we need to analyze sewings is summarized in the following.

LEMMA 1.6.6. *Let M_0 be a compact, connected Seifert fibered space with no exceptional fibers and whose boundary consists of a single torus C . If M_0 is orientable, then there is only one simple closed crossing curve on C up to isotopy and reversal that is the boundary of a section for the orbit surface of M_0 . If M_0 is non-orientable, then there are infinitely many simple closed crossing curves on C that are boundaries of sections of the orbit surface, and given any one such curve J_1 , then any crossing curve J_2 on C is the boundary of a section if and only if it differs from J_1 by an even multiple of the fiber in which case there is a fiber preserving self homeomorphism of M_0 that carries J_1 to J_2 or its reverse.*

PROOF The last provision can be deduced from Lemma 1.3.5. ■

We return to M , a closed, connected Seifert fibered space with no exceptional fibers. What we have to say about M will depend on the orientation of M if M is orientable. Thus we assume that M is oriented or that M is non-orientable. Let N be a fibered solid torus neighborhood of some fiber and let $M_0 = M - \overset{\circ}{N}$. Let Q be a boundary of a section of the orbit surface of M_0 and let m be a meridian of N . Both Q and m lie on $\partial M_0 = \partial N$ and we can look at $\psi(m, Q)$.

First assume that M is oriented. Let N inherit the orientation, and let ∂N be oriented consistently with N . Both Q and m are crossing curves (since N is

an ordinary solid torus), so for a fiber H on ∂N , both (H, Q) and (H, m) are valid generating pairs for $H_1(\partial N)$. Pick an orientation for H . The pairing $\psi(\cdot, \cdot)$ is determined by the orientation of ∂N . There are unique orientations on Q and m so that $\psi(H, Q) = \psi(H, m) = +1$. Let $b = \psi(m, Q)$. Using the generating pair (H, Q) for $H_1(\partial N)$, this expresses m as

$$(1.2) \quad m = \psi(m, Q)H + \psi(H, m)Q = bH + Q.$$

If M is non-orientable, then chose an orientation for N arbitrarily and use the rest of the steps above to calculate $b = \psi(m, Q) \pmod 2$ given as either $+1$ or 0 .

Various choices were made in the calculation of b . If the opposite choice of orientation is made for H , then the requirements on Q and m will force simultaneous reversal of orientations on them and we will preserve $\psi(m, Q)$. If a different fiber H' and fibered solid torus neighborhood N' are chosen for the drilling, then Theorem 1.4.6 gives an (orientation preserving), fiber preserving self homeomorphism of M carrying N to N' . This will carry a section of $M - \mathring{N}$ to a section of $M - \mathring{N}'$ and a meridian of N to meridian of N' (in an orientation preserving way if an orientation is present). This will also preserve $\psi(m, Q)$. If M is not orientable, then the opposite choice of orientation for N simply reverses the sign of b which makes no difference mod 2. From this it is clear that we get the same calculation for b from any Seifert fibered space that has the same (oriented) fiber type as M .

Consider the triple (G, ϕ, b) associated with M where G is the orbit surface, $\phi : \pi_1(G) \rightarrow \mathbf{Z}_2$ is the classifying homomorphism and b is as calculated above. This triple is a well defined invariant of the (oriented if applicable) fiber type of M . We now wish to know whether the triple distinguishes (oriented if applicable) fiber types.

Let M and M' be closed, connected Seifert fibered spaces without exceptional points that are oriented or non-orientable and let them have equivalent triples (G, ϕ, b) and (G', ϕ', b') in that ϕ and ϕ' are equivalent and $b = b'$. Since the orientability of a Seifert fibered space is determined by its classifying homomorphism, we know that M and M' are either both oriented or both non-orientable. Drill out a fiber from each of M and M' to create M_0 and M'_0 respectively. Since the classifying homomorphisms are equivalent, there is a fiber preserving homeomorphism h from M_0 to M'_0 .

First assume that the spaces are orientable. We can arrange that h be orientation preserving since by Corollary 1.6.3.3, there is an orientation reversing, fiber preserving homeomorphism from M_0 to itself. Choose an orientation for a fiber H in ∂M_0 and orient $h(H)$ consistently. Orient ∂M_0 and $\partial M'_0$ consistently with the orientations of the ordinary solid tori that were removed from M and M' to form M_0 and M'_0 . This forces orientations on Q , m , Q' and m' with notation continuing the pattern above so that h carries Q to a curve homologous to Q' and m to a curve homologous to m' . Now M is obtained from M_0 by sewing an ordinary solid torus to ∂M_0 with meridian going to $Q + bH$ and M' is obtained from M'_0 by sewing an ordinary solid torus to $\partial M'_0$ with meridian going to $Q' + bH'$. But h carries $Q + bH$ to a curve homologous to $Q' + bH'$ and from Theorem 1.3.8 we know that there is a homeomorphism from M to M' that extends h .

Assume that the spaces are non-orientable. Now the curves Q and Q' used in computing b and b' may not be related by having $h(Q)$ homologous to Q' . However, they are known to differ by an even multiple of the fiber and Lemma 1.6.5 gives a fiber preserving, self homeomorphism of M_0 which can be used to alter h so that $h(Q)$ is homologous to Q' . We finish the argument as in the orientable case. We have shown.

THEOREM 1.6.7. *Let M be a closed, connected Seifert fibered space with no exceptional fibers, either oriented or non-orientable. Then the triple (G, ϕ, b) is a well defined invariant of the (oriented if applicable) fiber type of M that completely determines the (oriented if applicable) fiber type of M . ■*

We refer to the integer b as the obstruction to a section for the orbit surface.

We can now enumerate the closed, connected Seifert fibered spaces with no exceptional fibers. There are four possible combinations of the orientability of the Seifert fibered space and its orbit surface. We indicate these by (O, o) , (O, n) , (N, o) and (N, n) . The first (upper case) letter refers to the orientability (O) or non-orientability (N) of the Seifert fibered space. The second (lower case) letter refers to orbit surface. For a given surface G there is only one class of Seifert fibered spaces over G that gives the combinations (O, o) , (O, n) and (N, o) . This is because there are only two classes of classifying homomorphisms for an orientable orbit surface, one resulting in orientable Seifert fibered spaces, and the other in non-orientable spaces, and there is only one class of homomorphisms, the one containing the orientation homomorphism ρ , that give orientable Seifert fibered spaces over a non-orientable orbit surface. For a given non-orientable surface, there are up to three classes that give (N, n) . We distinguish these by labeling them (N, n, I) , (N, n, II) and (N, n, III) . The first uses the class of the trivial homomorphism from $\pi_1(G)$ to \mathbf{Z}_2 , the second and third use the non-trivial homomorphisms that disagree with ρ where the second takes the unique involution of $H_1(G)$ to -1 , and the third takes it to $+1$. Since an orientable surface is determined by its genus g (number of handles) and a non-orientable surface is determined by the number k of crosscaps, we have a complete list of classes in the symbols (O, o, g) , (O, n, k) , (N, o, g) , (N, n, I, k) , (N, n, II, k) and (N, n, III, k) . There are some restrictions on g and k . For example there is no class $(N, n, III, 2)$ since the Klein bottle does not have enough crosscaps to

admit the required homomorphism. Within each class there are certain closed, connected Seifert fibered spaces with no exceptional fibers. These are identified up to (oriented if applicable) fiber type by $(O, o, g | b)$, $(O, n, k | b)$, $(N, o, g | b)$, $(N, n, I, k | b)$, $(N, n, II, k | b)$ and $(N, n, III, k | b)$ where $b \in \mathbf{Z}$ if the Seifert fibered space is oriented and $b \in \{0, 1\}$ if it is non-orientable. Note that for a given closed, connected orbit surface, there are only finitely many (up to a maximum of 6) fiber types of non-orientable Seifert fibered spaces with no exceptional fibers.

1.6.4. Crossing curves of fibered solid tori. We will analyze Seifert fibered spaces with exceptional fibers by relating them to spaces in the same class with no exceptional fibers. From a space M with an exceptional fiber, we can create a space M' with one fewer exceptional fiber by drilling out the exceptional fiber and replacing it with an ordinary fiber. That is, a fibered solid torus neighborhood N of the exceptional fiber is replaced by an ordinary solid torus T . What M and M' have in common is the space $M_0 = M - \mathring{N} = M' - \mathring{N}'$. To go in a well defined manner from M to M' , we need to be able to tell where the meridian of N' should go on ∂M_0 . This is a crossing curve since N' is an ordinary solid torus. To go in a well defined manner from M' to M , we need to be able to tell where the meridian of N should go on ∂M_0 . Thus we need a way of starting with a fibered solid torus N and picking out a well defined crossing curve and a technique for recovering the original meridian after the fibered solid torus has been forgotten. This is the subject of the current section. We will need to apply what we say to both orientable and non-orientable manifolds, so we consider both settings here.

Let N be a fibered solid torus. We assume first that N is oriented. Let m be a meridian arbitrarily oriented. Let H be a fiber in ∂N , oriented so that $\mu = \psi(H, m)$ is positive. (Recall that $\psi(H, m)$ can never be zero.) In what

follows we do not need to refer to a longitude, so we avoid it. A crossing curve Q for ∂N will intersect H once, so H and Q can be used to generate $H_1(\partial N)$. Since N is oriented, we get an orientation on ∂N . We can use this orientation to declare that certain orientations on H and Q are consistent with the orientation. This condition can be captured by the equality $\psi(H, Q) = +1$. If one crossing curve Q' is found so that $\psi(H, Q') = +1$, then all other crossing curves Q'' with $\psi(H, Q'') = +1$ take the form $Q'' = Q' + nH$, $n \in \mathbf{Z}$. Thus all such crossing curves Q'' have $\psi(m, Q'')$ defined up to integral multiples of μ . There is then a unique crossing curve Q and a unique integer β with $\psi(H, Q) = +1$ and $\beta = \psi(m, Q)$ with $0 \leq \beta < \mu$. If m were chosen with the opposite orientation, then simultaneously reversing the orientation of H and Q would then satisfy all the conditions above with the same integer β . The only choice made in the above discussion was the orientation of m , so N and its orientation determine Q up to reversal and determine β completely.

Now assume that the orientation on ∂N is known, that H and Q are known without fixed orientations, and that μ and β are known. If we demand that $\psi(H, Q) = +1$, then there are two possible choices for the orientations of H and Q as a pair, and these differ from each other by simultaneous reversal. These two choices are identical to the two orientation arrangements discussed in the previous paragraph. Thus setting $m = \beta H + \mu Q$ recovers m up to reversal of orientation.

We thus have for oriented N that H and Q are determined up to orientation and the pair (μ, β) is uniquely determined. Further, the orientation on ∂N , the unoriented curves H and Q and the pair (μ, β) determine m up to orientation.

What has actually taken place in the previous paragraphs is that H and Q can be expressed in terms of a meridian-longitude generating pair (m, l) for $H_1(\partial N)$ as the columns of $\begin{pmatrix} \nu & \alpha \\ -\mu & \beta \end{pmatrix}$, and m and l can be expressed in terms of the

generating pair (H, Q) as the columns of the inverse matrix $\begin{pmatrix} \beta & -\alpha \\ \mu & \nu \end{pmatrix}$ which expresses m as $m = \beta H + \mu Q$.

If N is not oriented, then H and the set of possible crossing curves Q'' are still the same, but there is no way to choose among the possible orientations. The set of intersection numbers $\psi(m, Q'')$ are ambiguous as to sign. Thus m , H and the crossing curves are oriented arbitrarily and we define μ as $|\psi(H, m)|$ and β as the minimum of $|\psi(m, Q'')|$ over all crossing curves Q'' . This gives $0 \leq \beta \leq \mu/2$. For $\mu > 2$, there will only be one crossing curve Q , up to change in orientation, that achieves this minimum. For $\mu = 2$, Q is not determined. In terms of a generating meridian-longitude pair, we can have $H = \begin{pmatrix} 1 \\ -2 \end{pmatrix}$ with both $Q = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ and $Q = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$ giving $\psi(H, Q) = 1$ and $|\psi(m, Q)| = 1$. This ambiguity is unavoidable and will be shown later not to be important.

Now assume that ∂N has no fixed orientation, that H and Q are known, that the pair (μ, β) is known, and that $\mu > 2$. We can express m as $m = \beta H + \mu Q$. There are four combinations of orientations of H and Q that break into two groups of two combinations. In a given group, one combination is obtained from the other by simultaneous reversal of H and Q . Within each group m is determined up to reversal. However, the two groups give different curves for m . Up to sign, the two possibilities for m are represented by $\pm\beta H + \mu Q$.

Thus for unoriented N with centerline of index at least 3, we get uniquely determined Q , up to reversal, and uniquely determined (μ, β) . These determine two possible curves, up to orientation, for m . We will also see later that this ambiguity is not important.

For a fibered solid torus N , oriented or not, we call (μ, β) the "*crossing invariants determined by*" N . We call Q the "*crossing curve determined by*" N except when N is unoriented and its centerline is a fiber of index 2.

1.6.5. Closed, oriented Seifert fibered spaces. If M is a closed, connected, oriented Seifert fibered space, then there is a finite set $\{H_1, H_2, \dots, H_n\}$ of exceptional fibers. Let $\{N_1, N_2, \dots, N_n\}$ be pairwise disjoint fibered solid torus neighborhoods of the exceptional fibers. Let $\{Q_1, Q_2, \dots, Q_n\}$ be the crossing curves determined by the neighborhoods, and let $(\mu_1, \beta_1, \dots, \mu_n, \beta_n)$ be the crossing invariants. We can assume that the H_i have been numbered so that the μ_i are in non-decreasing order, and the β_i in a group of constant μ_i are in non-decreasing order. The N_i can be removed and replaced by ordinary solid tori $\{N'_1, N'_2, \dots, N'_n\}$ where the sewing is determined by requiring that a meridian of each N'_i be sewn to Q_i . This gives a space M' with no exceptional fibers. Note that the invariance of fibered solid torus neighborhoods of fibers and the invariance of the crossing curves shows that M' is completely determined up to oriented fiber type by M . The space M' is determined up to oriented fiber type by one of the symbols $(O, o, g \mid b)$ or $(O, n, k \mid b)$. We can thus associate one of the following symbols with M :

$$(O, o, g \mid b, \mu_1, \beta_1, \dots, \mu_n, \beta_n)$$

or

$$(O, n, k \mid b, \mu_1, \beta_1, \dots, \mu_n, \beta_n).$$

The symbol is completely determined by M .

If one of the symbols above is given, then we build a realization. We start with an oriented space M' with no exceptional fibers determined by $(O, o, g \mid b)$ or $(O, n, k \mid b)$. We chose n fibers and pairwise disjoint ordinary solid torus neighborhoods N'_i of these fibers. Each N'_i has a meridian Q_i . The orientation that N'_i inherits from M' determines an orientation on $\partial N'_i$. The orientation on $\partial N'_i$, a fiber H_i in $\partial N'_i$, and the meridian Q_i determine a curve $M_i = \beta_i H_i + \mu_i Q_i$ up to orientation. There is a fibered solid torus N_i that can be sewn to $\partial N'_i$ in a way that preserves fibers and takes a meridian to m_i . The resulting space is

a realization of the symbol. We argue that any two realizations have the same oriented fiber type. The oriented fiber type of M' is determined. If different fibers and neighborhoods are chosen for the N'_i then the transitivity of fibers will give an orientation preserving, fiber preserving homeomorphism that carries one set of neighborhoods to the other. It will also carry one set of meridians to the other. The curves for attaching the new meridians are determined up to reversal and Theorem 1.3.8 guarantees that the resulting space is determined up to oriented fiber type. If the given symbol came from an oriented space M , then M is a realization because of the way that the symbol was derived from M . Thus all realizations are of the same oriented fiber type of M . We have the following.

THEOREM 1.6.8. *Let M be a closed, connected, oriented Seifert fiber space. Then the appropriate symbol*

$$(O, o, g \mid b, \mu_1, \beta_1, \dots, \mu_n, \beta_n)$$

or

$$(O, n, k \mid b, \mu_1, \beta_1, \dots, \mu_n, \beta_n)$$

is a well defined invariant of the oriented fiber type of M that completely determines the oriented fiber type of M . ■

1.6.6. Closed, non-orientable Seifert fibered spaces. Let M be a closed, connected, non-orientable Seifert fibered space. As in the previous section there is a finite set $\{H_1, \dots, H_n\}$ of exceptional fibers with corresponding fibered torus neighborhoods $\{N_1, \dots, N_n\}$. The possibility that some of the fibers are of index 2 changes the analysis, so we start by assuming that there are none. Thus we get a corresponding set $\{Q_1, \dots, Q_n\}$ of crossing curves determined by the N_i and crossing invariants $(\mu_1, \beta_1, \dots, \mu_n, \beta_n)$ ordered as in the previous section. As before, we can remove each N_i and replace it with an ordinary solid torus N'_i

with meridian sewn to Q_i . This creates a space M' with no exceptional fibers. This space is determined by one of the symbols $(N, o, k | b)$, $(N, n, \text{I}, k | b)$, $(N, n, \text{II}, k | b)$, or $(N, n, \text{III}, k | b)$. We then associate one of

$$\begin{aligned} &(N, o, k | b, \mu_1, \beta_1, \dots, \mu_n, \beta_n), \\ &(N, n, \text{I}, k | b, \mu_1, \beta_1, \dots, \mu_n, \beta_n), \\ &(N, n, \text{II}, k | b, \mu_1, \beta_1, \dots, \mu_n, \beta_n), \end{aligned}$$

or

$$(N, n, \text{III}, k | b, \mu_1, \beta_1, \dots, \mu_n, \beta_n)$$

with M . As before, the symbol is determined by M .

Given a symbol as above, we build a realization in a well defined manner as before up to the point of adding a new solid torus where an ordinary solid torus N'_i was before. We are faced with two possible curves $m = \pm\beta_i H_i + \mu_i Q_i$ to chose from. However, it follows from Lemma 1.6.5 that there is a fiber preserving, orientation reversing map on $M' - N'_i$ that takes Q_i to $\pm Q_i$ and H_i to $\mp H_i$. This is seen to take one possibility for m to the other or its reverse. Thus the two possible ways of attaching N_i are seen to yield spaces of the same fiber type.

If there are fibers of index 2 present, then our ordering of the crossing invariants has $\mu_i = 2$ for all i less than s where s is the number of exceptional fibers of index 2. All α_i , $i \leq s$, equal 1. The crossing curves Q_i for $i > s$ are determined. There is a problem in choosing crossing curves for $i \leq s$. This will create no problem in recovering M as is shown by the next lemma.

LEMMA 1.6.9. *Let M_0 be a non-orientable Seifert fibered space with at least one torus boundary component C . Let N be a fibered solid torus of type 1/2 and assume that it is possible to obtain M from M_0 by attaching N to M_0 with a fiber preserving homeomorphism h from ∂N to C . Then the fiber type of M is independent of which fiber preserving homeomorphism h is used.*

PROOF To give a frame of reference for the duration of the proof, we orient C and ∂N arbitrarily. We know there is a crossing curve Q on ∂N for which $m = 2Q + H$. Since the sewings of ∂N to C must be fiber preserving, the sewings are distinguished by what the sewing maps do on Q and what the maps do to the orientation of the fibers. The possible images of Q are all crossing curves on C , and these are all of the form $\pm Q' + nH$, $n \in \mathbf{Z}$, where Q' is one fixed crossing curve on C . We have already seen that reversing the effect of the orientations on either Q or H has no effect on the result of the sewing. Thus we can consider only curves of the form $Q' + nH$, $n \in \mathbf{Z}$, and assume that h preserves the orientations of the fiber. We also know from Lemma 1.6.5 that curves that differ by even multiples of the fiber give equivalent results. We can thus consider only Q' and $Q' - H$ as the possible images of Q . The images of m under these two sewings are $2Q' + H$ and $2Q' - H$. But again Lemma 1.6.5 gives a fiber preserving, self homeomorphism of M_0 that fixes Q' and H up to sign and reverses the orientation on C . This carries one image of m to the other or its reverse and the resulting spaces are fiber homeomorphic. ■

There is thus a lack of ambiguity in determining M from another space. However, fibers of index 2 do not determine crossing curves and we have an ambiguity in determining another space from M . To get around this we will not drill out the fibers of index 2. This leads to the following general procedure.

Let M be a closed, non-orientable Seifert fibered space. Let s be the number of exceptional fibers of index 2, let $\{H_1, \dots, H_n\}$ be the exceptional fibers of index greater than 2, let $\{N_1, \dots, N_n\}$ be pairwise disjoint fibered solid torus neighborhoods of the H_i , let $\{Q_1, \dots, Q_n\}$ be the crossing curves determined by the H_i , and let $(\mu_1, \beta_1, \dots, \mu_n, \beta_n)$ be the crossing invariants of the H_i ordered in the usual way. We create a space by removing all the N_i and sewing in ordinary fibered solid tori N'_i with meridians going to the curves Q_i . This creates a space

with s exceptional fibers of index 2. We denote this space by M_s . We can drill out the fibers of index 2 from M_s to create a space M_0 . Since M_0 has boundary, it is determined by its class and thus by M . Since it is non-orientable, it is determined by one of the symbols $(N, o, g \mid -)$, $(N, n, I, k \mid -)$, $(N, n, II, k \mid -)$ or $(N, n, III, k \mid -)$. Since sewing in the s exceptional fibers of index 2 gives a space that is independent of the sewings, we have that M_s is dependent only on M and not on any choices. We now associate one of the following with M :

$$\begin{aligned} &(N, o, k \mid (b, s), \mu_1, \beta_1, \dots, \mu_n, \beta_n), \\ &(N, n, I, k \mid (b, s), \mu_1, \beta_1, \dots, \mu_n, \beta_n), \\ &(N, n, II, k \mid (b, s), \mu_1, \beta_1, \dots, \mu_n, \beta_n), \end{aligned}$$

or

$$(N, n, III, k \mid (b, s), \mu_1, \beta_1, \dots, \mu_n, \beta_n)$$

where s is the number of exceptional fibers of index 2 and b is irrelevant if $s > 0$. The determination of the symbol by M has been discussed above. The determination of M by the symbol follows from the fact that the symbol determines uniquely the space M_s and the numbers $(\mu_1, \beta_1, \dots, \mu_n, \beta_n)$ determine M from M_s . We have the result summarized as the following.

THEOREM 1.6.10. *Let M be a closed, connected, non-orientable Seifert fiber space. Then the appropriate symbol*

$$\begin{aligned} &(N, o, k \mid (b, s), \mu_1, \beta_1, \dots, \mu_n, \beta_n), \\ &(N, n, I, k \mid (b, s), \mu_1, \beta_1, \dots, \mu_n, \beta_n), \\ &(N, n, II, k \mid (b, s), \mu_1, \beta_1, \dots, \mu_n, \beta_n), \end{aligned}$$

or

$$(N, n, III, k \mid (b, s), \mu_1, \beta_1, \dots, \mu_n, \beta_n)$$

is a well defined invariant of the fiber type of M that completely determines the fiber type of M . ■

1.6.7. Compact Seifert fibered spaces with boundary. Compact Seifert fibered spaces with boundary have been discussed to some extent above. We do not include a complete discussion of exceptional fibers here, but we include one lemma that shows that even in the orientable case, there needs to be less information kept about the location of crossing curves. The use of the following is when C is to be used as a place to attach a fibered solid torus.

LEMMA 1.6.11. *Let M be a Seifert fibered space with at least two boundary components C and C' with C a torus. Let Q and Q' be two crossing curves for C . Then there is a fiber preserving, self homeomorphism of M that is fixed on all boundary components of M except C and C' that carries Q to Q' or its reverse.*

PROOF Let G be the orbit surface and let J and J' be the images of C and C' . Let α be an arc from J to J' that avoids exceptional points. We can split G along α and we can split M to get M_- along the annulus A that is the preimage of α . We get two copies A' and A'' of A in the splitting. There is a fiber preserving isotopy of M_- (in the strong sense in that each fiber is mapped to itself throughout the isotopy) that rotates the fibers of A' through one full circle and that only moves points in a small neighborhood of A' . The end of this isotopy gives a fiber preserving self homeomorphism of M that carries a given crossing curve to itself plus one copy of the fiber. Repetitions of this suffice to carry any crossing curve to any other or its reverse. ■

1.6.8. The diagram of a closed Seifert fibered space. The “symbol” associated with a closed, connected Seifert fibered space has the shape (\mid) where the information to the left of the vertical bar determines the class of the space, and the information to the right of the bar determines the fiber type of the space within that class. The determination of the class of the space can also be made with a geometric object — the classifying space of the class — the unique space in

the class with no exceptional fibers and exactly one torus boundary component. We can also associate a geometric object with the information to the right of the vertical bar. We will call this object a "*diagram*" associated with a closed, connected Seifert fibered space.

Let M be a closed, connected Seifert fibered space that is either oriented or non-orientable. Let $\{H_1, \dots, H_n\}$ be the set of exceptional fibers and let $\{N_1, \dots, N_n\}$ be pairwise disjoint fibered solid torus neighborhoods of the H_i . Let $(\mu_1, \beta_1, \dots, \mu_n, \beta_n)$ be the crossing invariants ordered in the usual way. Let s be the number of μ_i equal to 2. If $s \neq 0$ and M is not orientable, then crossing curves for the corresponding ∂N_i are not determined. Let $\{Q_1, \dots, Q_n\}$ be crossing curves determined by the N_i where possible and chosen arbitrarily where not. Let M' be obtained from M by replacing the N_i by ordinary solid tori $\{N'_1, \dots, N'_n\}$ so that a meridian of each N'_i is sewn to Q_i . We will remember where the meridian m_i of N_i is on $\partial N'_i$. Let N_0 be a fibered solid torus in M' that contains all the N'_i in its interior. Note that $M_0 = M' - \overset{\circ}{N}_0$ is the classifying space for the class of M . Our diagram for M will consist of the fibered space $V = N_0 - \bigcup \overset{\circ}{N}'_i$ together with certain curves on the boundary components of V and (if M is oriented) an orientation of V . The fiber structure of V is simply the product of an n punctured disk with S^1 . On each $\partial N'_i = \partial N_i$ we will preserve m_i the meridian of N_i . On $\partial N_0 = \partial M_0$ we will preserve a curve Q_0 that is the boundary of a section for M_0 . This will be ambiguous if M is non-orientable. Note that Q_0 is a crossing curve on ∂N_0 for the fiber structure. If M is oriented, then M and M' share the space M_0 (they actually share more) so M_0 and thus M' inherits an orientation from M . Then N_0 and thus V inherit an orientation from M' . The diagram is the fibered space V with its orientation, if available, together with the curves Q_0 and the m_i .

We now need to argue that V recovers all the information in the symbol for M that is to the right of the vertical bar. That is, the numbers b , s , the μ_i

and the β_i . We will see that the ambiguity in Q_0 will not interfere with these determinations.

By Theorem 1.3.4, there is only one fiber type of fibered solid torus that can be sewn by a fiber preserving homeomorphism of the boundary to $\partial N'_i$ so that a meridian is sewn to m_i . This fibered solid torus (which we may as well refer to as N_i) can be oriented consistently with V (if an orientation of V is available). This determines crossing invariants (μ_i, β_i) and sometimes a crossing curve Q_i in the usual way. This gives us the number s of exceptional fibers of index 2, the μ_i and the β_i . We only need b , the obstruction to the section.

The crossing curve Q_i is determined up to orientation unless $s \neq 0$ and V is non-orientable in which case Q_i is ambiguous. However, when V is non-orientable and $s \neq 0$, there is no need to determine b . We thus assume that V is oriented, or that $s = 0$.

Assume first that V is oriented. If an orientation of an arbitrary fiber H in V is chosen, then all fibers can be oriented consistently with H . Then the orientation of each crossing curve Q_i is determined by requiring that $\psi(H_i, Q_i) = +1$ where H_i is a fiber in ∂N_i and ∂N_i is oriented to agree with the orientation on N_i . The curves Q_i are the meridians of the ordinary solid tori N'_i . If the N'_i are restored by adding them to V , then N_0 is recovered. We orient ∂N_0 consistently with N_0 . Since m_0 and Q_0 are crossing curves on ∂N_0 , we can orient m_0 and Q_0 by requiring that $\psi(H_0, m_0) = \psi(H_0, Q_0) = +1$ for some fiber H_0 in ∂N_0 . The orientations used on the ∂N_i were inherited from the N_i and are the reverse of the orientations that they would inherit from V . Since m_0 is a meridian for N_0 in which the N'_i are solid tori with meridians Q_i , we have $m_0 = \sum Q_i$ in $H_1(V)$. However, m_0 and Q_0 were oriented as required to determine b as $\psi(m_0, Q_0)$ and, from 1.2, determine m_0 as $bH_0 + Q_0$. Thus $\sum Q_i = bH_0 + Q_0$ and we have shown that $-Q_0 + \sum Q_i$ is homologous in V to a multiple of a fiber. The number b

is that multiple. If the opposite orientation is chosen for an arbitrary fiber H , then all orientations of Q_0 , m_0 and the Q_i reverse and the calculations remain the same.

If M is non-orientable and $s = 0$, then the curves Q_i and m_0 are still determined, but their orientations are not. Also, Q_0 is determined only up to an even multiple of the fiber. However, b is still determined mod 2 and this is all we need.

We illustrate a use for the diagram by showing the effect of reversal of orientation on the invariants of a closed, connected, oriented Seifert fibered space M . Let $-M$ denote the space M endowed with the opposite orientation of M . Let the M be determined by

$$(O, o, g \mid b, \mu_1, \beta_1, \dots, \mu_n, \beta_n)$$

or

$$(O, n, k \mid b, \mu_1, \beta_1, \dots, \mu_n, \beta_n).$$

Reversing the orientation of M changes neither the orbit surface nor the classifying homomorphism. We must look at the effects on the numbers to the right of the vertical bar. If V , m_i and Q_0 make up the diagram for M , then we obtain the diagram for $-M$ by reversing the orientation of V to get $-V$. If a fiber H was used to calculate the invariants for M , then it will be convenient to use the fiber $-H$ to calculate the invariants b' , μ'_i and β'_i for $-M$.

Using $-H$ instead of H means that we can still use the orientations on the m_i that were used for M . We have $\mu'_i = \psi(-H_i, m_i) = \mu_i$ since ψ is the negative of the intersection pairing used with M . If Q_i is a crossing curve appropriate for M , we must determine how to alter it to give a curve Q'_i appropriate for $-M$. We have $Q'_i = Q_i + xH_i$. This is correct since in $-V$ we get the correct intersection

with $-H$ without reversing the direction of Q_i . We want $\beta'_i = \psi(m_i, Q'_i)$ or the coefficient of $-H_i$ in m_i to be in the interval $[0, \mu_i)$. But

$$\begin{aligned} m_i &= \beta_i H_i + \mu_i Q_i \\ &= \beta_i H_i + \mu_i (Q'_i - x H_i) \\ &= (\beta_i - \mu_i x) H_i - \mu_i Q'_i \\ &= (\mu_i x - \beta_i) (-H_i) - \mu_i Q'_i \end{aligned}$$

so that we get the the right value when $x = 1$ and $\beta'_i = \mu_i - \beta_i$.

We need to determine b' . We have $Q'_0 = Q_0$ since we have replaced H by $-H$. We determine b' from the equality $-Q'_0 + \sum Q'_i = b'(-H)$. The left side is $-Q_0 + \sum(Q_i + H_i) = -Q_0 + nH + \sum Q_i$ since all fibers are homologous in V . But this is $bH + nH = (-n - b)(-H)$ so $b' = -n - b$. Note that the functions $f_i(\beta_i) = \mu_i - \beta_i$ and $f(b) = -n - b$ have period two. We have the following.

THEOREM 1.6.12. *Let M be a closed, connected, oriented Seifert fibered space determined by either*

$$(O, o, g \mid b, \mu_1, \beta_1, \dots, \mu_n, \beta_n)$$

or

$$(O, n, k \mid b, \mu_1, \beta_1, \dots, \mu_n, \beta_n).$$

Then the Seifert fibered space with the opposite orientation is detmined by

$$(O, o, g \mid -n - b, \mu_1, \mu_1 - \beta_1, \dots, \mu_n, \mu_n - \beta_n)$$

or

$$(O, n, k \mid -n - b, \mu_1, \mu_1 - \beta_1, \dots, \mu_n, \mu_n - \beta_n).$$

■

CHAPTER 2

Topology of Seifert fibered spaces

Most of this chapter is based on pages 83–101 of Chapter VI of W. Jaco’s book “Lectures on Three-manifold topology,” Regional Conference Series in Mathematics, number 43, AMS, Providence, 1980. Many of the details were taken from Chapter II, Sections 3 and 4 of W. Jaco and P. Shalen: “Seifert fibered spaces in 3-manifolds,” Mem. Amer. Math. Soc., number 220, (1979). The material on covers of Seifert fibered spaces is taken from Section 9 of Seifert’s paper, identified in the opening paragraph of Chapter 1.

We wish to classify Seifert fibered spaces up to homeomorphism. We need invariants that do not depend on the fibers, but that give topological information. The most powerful invariant is the fundamental group. In this chapter, we calculate the fundamental group of Seifert fibered spaces, and use the results of the calculation to gather information about the spaces. In particular we will classify the compact, connected Seifert fibered spaces up to homeomorphism. [The course ended before this was done. Compact Seifert fibered spaces with non-empty boundary were classified.] Along the way we will figure out which Seifert fibered spaces are aspherical, have incompressible boundary and have an incompressible surface.

We need some standard concepts from the topology of 3-manifolds. We know that a 3-manifold is irreducible if every embedded 2-sphere bounds a 3-cell. We say that a 3-manifold is “ P^2 -irreducible” if it is irreducible and it contains no embedded 2-sided projective plane. The projective plane theorem of Epstein, a

generalization of the sphere theorem, says that a 3-manifold with non-trivial π_2 fails to be P^2 -irreducible. A P^2 -irreducible 3-manifold has trivial π_2 , and if it has non-empty boundary or infinite π_1 , then its universal cover has trivial π_1 , trivial π_2 and trivial H_n for all $n \geq 3$. Thus a P^2 -irreducible 3-manifold with infinite π_1 or with non-empty boundary is aspherical.

A surface S is "*properly embedded*" in a 3-manifold M if it is embedded and $\partial S = S \cap \partial M$. A properly embedded surface S is "*boundary parallel*" in M if there is an embedding of $S \times I$ into M carrying $S \times \{0\}$ onto S and $(\partial S \times I) \cup (S \times \{1\})$ into ∂M . A surface in M is said to be "*incompressible*" if either it is a 2-sphere bounding no 3-cell, or is properly embedded, two sided and not a disk and the inclusion induces an injection on π_1 , or if it is a properly embedded disk that is not boundary parallel, or if it is embedded in the boundary and is not a disk and the inclusion into M induces an injection on π_1 . The technicalities are to avoid certain trivial situations.

We say that a 3-manifold whose boundary components are all incompressible is "*boundary irreducible*". This is satisfied vacuously if the boundary of the manifold is empty. We say that a properly embedded, two sided surface S in a 3-manifold M is "*boundary compressible*" if there is a disk D in M so that $D \cap S$ is an arc A in ∂D , $D \cap \partial M$ is an arc B in ∂D with $\partial D = A \cup B$ and so that A is not boundary parallel in S . (Shift the definition of boundary parallel down one dimension.) A properly embedded surface is "*boundary incompressible*" if it is not boundary compressible. Unfortunately, boundary irreducible and boundary incompressible were named independently and have stuck.

A compact, orientable, irreducible 3-manifold with an incompressible surface is called a "*Haken*" manifold. We will try to have all our statements apply to both orientable and non-orientable 3-manifolds, but this will not be possible. Towards the end of this chapter, we will restrict ourselves to orientable manifolds.

We are now ready to study some of these properties. Seifert fibered spaces with boundary are easier to handle than closed Seifert fibered spaces, and we start with some elementary observations about bounded spaces.

2.1. Bounded, compact Seifert fibered spaces

LEMMA 2.1.1. *Let T be a fibered solid torus. Then a fiber of T represents a non-trivial element of $\pi_1(T)$.*

PROOF Let T be determined by $\nu/\mu \pmod 1$ with ν and μ in reduced terms and with $\mu > 0$. Then the centerline of T represents a generator of $\pi_1(T)$, and any other fiber represents μ times the generator. ■

COROLLARY 2.1.1.1. *Let T be a fibered solid torus. Then a saturated annulus in ∂T is an incompressible surface in T .* ■

We need standard results about 3-manifolds which first need standard results from group theory. Let X be a simplicial complex and let A and B be pairwise disjoint subcomplexes that form connected, closed subsets of X . Let $h : A \rightarrow B$ be a homeomorphism. Let X_h be formed from X by using h to identify A with B . We make no assumptions on the connectivity of X , but do assume that X_h is connected. Thus X has at most two components X_1 and X_2 . We assume that $\pi_1(A)$ injects into the fundamental group of the component of X containing A and make a similar assumption about B . If X is connected, then we say that $\pi_1(X_h)$ is an HNN extension of $\pi_1(X)$ along $\pi_1(A)$ and $\pi_1(B)$. If X is not connected, then $\pi_1(X_h)$ is the free product with amalgamation of $\pi_1(X_1)$ and $\pi_1(X_2)$ (along the amalgamating subgroups $\pi_1(A)$ and $\pi_1(B)$). These constructions can be given purely algebraic definitions so that given isomorphic subgroups A and B (with a given isomorphism between them) of one or two groups G and H , we can form an HNN extension along A and B or a free product with amalgamation

along A and B depending on whether one or two parent groups are involved. We give the next lemma in topological terms although it can be given in algebraic terms.

LEMMA 2.1.2. *With X_h, X, X_1, X_2, A and B as above, we have the following. Each of $\pi_1(X), \pi_1(X_1), \pi_1(X_2)$, (whichever exist) and $\pi_1(A)$ and $\pi_1(B)$ inject into $\pi_1(X_h)$. Any torsion element of $\pi_1(X_h)$ is conjugate to an element of $\pi_1(X), \pi_1(X_1)$ or $\pi_1(X_2)$.*

PROOF See pages 178–187 of *Combinatorial Group Theory* by Lyndon and Schupp. ■

COROLLARY 2.1.2.1. *With the notation as above, $\pi_1(X_h)$ has torsion if and only if one of $\pi_1(X), \pi_1(X_1)$ or $\pi_1(X_2)$ has torsion.* ■

LEMMA 2.1.3. *Let M be a P^2 -irreducible 3-manifold, not necessarily connected. Let F and G be disjoint, connected, incompressible surfaces in ∂M not necessarily in different components of M and let h be a homeomorphism from F to G . Let M_h be the 3-manifold obtained by identifying each $x \in F$ to $h(x)$ and assume that M_h is connected.. Then M_h is P^2 -irreducible, the common image of F and G in M_h is incompressible in M_h , and the fundamental group of each component of M injects into $\pi_1(M_h)$.*

PROOF Because of the previous lemma, only the first conclusion needs proof. The proof of the other conclusions could also be done geometrically with similar techniques.

If S is a 2-sphere or 2-sided projective plane in M_h , then it can be put in general position with respect to the image of F and G in M_h that we continue to refer to as F . No circle of $S \cap F$ can be orientation reversing on S since it would be one sided on S and could not reside on the two sided surface F . Thus

every circle of $S \cap F$ bounds at least one disk on S . Since F is incompressible in M , a circle of $S \cap F$ that is innermost on S bounds a disk in S and a disk D in F that create a sphere whose only intersection with F is D . This sphere can be regarded as a subset of M where it bounds a 3-cell. An isotopy of S across this 3-cell lowers the number of components of intersection of S with F . Eventually S and F are disjoint and S is seen to bound a 3-cell in M_h if S is a sphere, or not exist if S is a 2-sided projective plane. ■

LEMMA 2.1.4. *Let M be a compact, connected Seifert fibered space with non-empty boundary. Then M is P^2 -irreducible, $\pi_1(M)$ is torsion free and each fiber of M represents a non-trivial element of $\pi_1(M)$.*

REMARK The trivial observation that $\pi_1(M)$ is infinite under the hypotheses of the lemma is worth making at this point.

PROOF There are pairwise disjoint arcs in the orbit surface so that the result of cutting along the arcs yields a set of disks with no more than one exceptional point in each. The spaces over these disks are fibered solid tori, and M is recovered from these tori by sewings along saturated annuli in their boundaries. However, a saturated annulus in the boundary of a fibered solid torus is incompressible in the fibered solid torus, and each solid torus is P^2 -irreducible. Thus M is P^2 -irreducible.

The fact that $\pi_1(M)$ is torsion free follows from the construction just given, from the fact that the fundamental group of a solid torus is torsion free and from the corollary above.

Note that each exceptional fiber has a multiple that is homotopic to an ordinary fiber. Also, any two ordinary fibers are homotopic. Since $\pi_1(M)$ is torsion free, we only have to show that one ordinary fiber is non-trivial. But this is true in a fibered solid torus, and the result follows from the construction just given and from Lemma 2.1.3. ■

The proof of the next lemma contains arguments that will be used repeatedly in the rest of the chapter.

LEMMA 2.1.5. *Let M be a compact, connected Seifert fibered space with a compressible boundary component. Then M is a fibered solid torus. In particular, M has at most one exceptional fiber.*

REMARK This lemma says that the only compact, connected Seifert fibered space that fails to be boundary irreducible is a fibered solid torus.

PROOF Let C be the compressible boundary component. Either C is a torus or a Klein bottle and its compression yields a 2-sphere. Since M is irreducible, the 2-sphere bounds a 3-cell and M is homeomorphic to a solid torus or a solid Klein bottle. In either case $\pi_1(M)$ is \mathbf{Z} . Let $p : M \rightarrow G$ be projection to the orbit surface. Since a fiber of M is non-trivial in $\pi_1(M)$ and p induces a surjection on π_1 , we have that $\pi_1(G)$ is torsion. But the only surface with boundary whose fundamental group is torsion is a disk. We will be done if we can show that there are fewer than two exceptional fibers.

We argue that it will suffice to show that if there are exactly two exceptional fibers then $\pi_1(M)$ cannot be cyclic. This will rule out the possibility that there are exactly two exceptional fibers. If there are at least three exceptional fibers, then there is an arc in G which splits G into a disk with exactly two exceptional points, and another disk with the rest of the exceptional points. The preimage of this arc is an incompressible annulus in M that separates M into two Seifert fibered spaces that are joined along the annulus. The fundamental groups of these two spaces inject into the fundamental group of M , and one space has two exceptional fibers and orbit surface a disk. Since we will show that the space with two exceptional fibers has non-cyclic π_1 we are done.

We now assume that there are two exceptional fibers. Let α be an arc in G that splits G into two disks with one exceptional point each. The annulus A over

α splits M into two fibered solid tori T_1 and T_2 with defining numbers ν_1/μ_1 and ν_2/μ_2 . Since the centerlines of these fibered solid tori are exceptional fibers, we have $\mu_1 > 1$ and $\mu_2 > 1$. The splitting gives two copies A_1 and A_2 of A . The fundamental group of A_i is carried by an ordinary fiber of T_i that represents μ_i times a generator of $\pi_1(T_i)$. The fundamental group of M is thus generated by two elements a and b and the only relation is $a^{\mu_1} = b^{\mu_2}$. Since each μ_i is at least 2, the group is a free product with amalgamation of two copies of \mathbf{Z} along a proper subgroup of each and is thus not cyclic. ■

COROLLARY 2.1.5.1. *Let M be a compact, connected Seifert fibered space, and let F be a two sided, saturated torus or Klein bottle in M containing no exceptional fibers and that is the preimage of a simple closed curve in the orbit surface that does not bound a disk with fewer than two exceptional points. Then F is incompressible in M .*

PROOF If false, then splitting M along F would result in a fibered solid torus, which contradicts the hypothesis. ■

REMARK There can be saturated Klein bottles that contain exceptional fibers and that are not saturated over simple closed curves. Consider the oriented I -bundle over the Klein bottle fibered by circles that contain orientation reversing curves of the Klein bottle. This has orbit surface a disk with two exceptional fibers of index 2. The image in the orbit surface of the saturated Klein bottle is an arc connecting the two exceptional points.

2.2. Fundamental groups of Seifert fibered spaces

We would like to extend the results above (where possible) to closed Seifert fibered spaces. The questions are a little more delicate especially since some of the generalizations are false. The 3-sphere is a Seifert fibered space in which all

fibers are trivial in the fundamental group. Also, $S^2 \times S^1$ is not irreducible. We need the power of the fundamental group.

We show first that results such as in the previous section can be obtained easily for a large class of Seifert fibered spaces.

LEMMA 2.2.1. *Let M be a closed, connected Seifert fibered space so that M either has an orientable orbit surface of genus at least one, or orbit surface S^2 with at least 4 exceptional fibers, or non-orientable orbit surface with at least 2 crosscaps, or orbit surface P^2 with at least 2 exceptional fibers. Then M is P^2 -irreducible, boundary irreducible and has an embedded incompressible torus or Klein bottle. It follows that M has infinite π_1 and is aspherical.*

PROOF If we can find an incompressible, saturated torus or Klein bottle F in M , then we can split M along F and use Lemma 2.1.4 to conclude that the result of the splitting is P^2 -irreducible. The incompressibility of F would then imply that M is irreducible. The other results would follow because the fundamental group of F is infinite and a P^2 -irreducible 3-manifold with infinite fundamental group is aspherical.

From Corollary 2.1.5.1, we get our surface F if there is a simple closed curve in the orbit surface that does not bound a disk with fewer than two exceptional points in it. The hypotheses that we are given have been tailored so as to guarantee the existence of such a curve. ■

Lemma 2.2.1 motivates much of what we do in this section. We first calculate the fundamental groups of Seifert fibered spaces, and then apply our calculation to some of the spaces singled out as problems by Lemma 2.2.1.

2.2.1. Calculating the fundamental group. We can use the realization construction to calculate the fundamental group of a Seifert fibered space. Let M be a compact, connected Seifert fibered space, let G be the orbit surface, let

$\{H_1, \dots, H_n\}$ be the exceptional fibers of M with pairwise disjoint fibered solid torus neighborhoods $\{N_1, \dots, N_n\}$, crossing curves $\{Q_1, \dots, Q_n\}$ and crossing invariants $(\mu_1, \beta_1, \dots, \mu_n, \beta_n)$. Let b be the obstruction to a section for the orbit surface if M is closed. Let ϕ be the classifying homomorphism. Since this is enough information to determine M , it should be enough to determine $\pi_1(M)$. Some of the crossing curves Q_i may be ambiguous, but since these ambiguities do not make M ambiguous, they will not make $\pi_1(M)$ ambiguous.

Let M' be obtained from M by replacing each N_i by an ordinary solid torus N'_i with meridian sewn to Q_i . Let M_0 be obtained from M' by removing the interiors of all the N'_i and the interior of an ordinary solid torus N_0 that is disjoint from the N'_i . We get M_0 from M by drilling out all the exceptional fibers and one ordinary fiber. The orbit surface G_0 of M_0 is a compact surface with boundary. It is obtained from G by removing $n + 1$ disks. We will label the boundary components of G_0 depending on their origin. We let $\{c_1, \dots, c_n\}$ be the components that arise from the removed N'_i . We let $\{d_1, \dots, d_m\}$ be the boundary components that arise from the boundary components of M . We let e be the boundary component that arises from the removal of N_0 . There are pairwise disjoint arcs $\{\alpha_1, \dots, \alpha_r\}$ in G_0 with boundaries in e so that if G_0 is cut along these arcs, then a single disk with holes results. If G_0 is orientable of genus g , then $r = 2g$. If G_0 is non-orientable with k crosscaps, then $r = k$. There are pairwise disjoint arcs $\{\alpha_{r+1}, \dots, \alpha_{r+m+n}\}$, disjoint from the arcs just mentioned, so that each c_i and each d_i is connected to e by one of these arcs. If G_0 is cut along all of the α_i , then a single disk E results. We recover G_0 by sewing together the copies of the arcs. We recover M_0 from the ordinary solid torus T over E by sewing together saturated annuli that are the pre-images of the copies of the arcs.

The fundamental group of G_0 is free. Each sewn arc corresponds to a free generator represented by a loop that pierces the arc once. If G_0 is orientable, then

the generators corresponding to $\{\alpha_1, \dots, \alpha_{2g}\}$ are usually grouped in pairs, a pair for each handle, and are denoted $\{a_1, b_1, \dots, a_g, b_g\}$. If G_0 is non-orientable, then the generators corresponding to $\{\alpha_1, \dots, \alpha_k\}$ are usually denoted $\{x_1, \dots, x_k\}$. The generators corresponding to the remaining $m + n$ arcs are represented by the boundary components c_i and d_i so we abuse notation and use $\{c_1, \dots, c_n\}$ and $\{d_1, \dots, d_m\}$ to denote the remaining generators. The fundamental group of G_0 is free on these generators, but it is useful to record how e relates to these generators. If G_0 is orientable, then

$$(2.1) \quad e = \prod [a_i, b_i] \prod c_i \prod d_i.$$

If G_0 is non-orientable, then

$$(2.2) \quad e = \prod x_i^2 \prod c_i \prod d_i.$$

The reconstruction of M_0 starts with the ordinary solid torus T . The fundamental group of T is free on one generator that we denote h since it is carried by any fiber. Each sewing of a pair of annuli results in another generator and a relation that comes from the fact that the annuli have non-trivial fundamental groups. We again abuse notation and label each generator with the same letter as the generator of $\pi_1(G_0)$ that is associated with the arc under the identified annuli. The identified annuli have fundamental groups generated by conjugates of h . One annulus can be thought of as being generated by h and the other can be thought of as being generated by h conjugated by the new generator. If the annuli are sewn to preserve the orientations of the fibers, then the conjugate is identified with h . Otherwise the conjugate is identified with h^{-1} . The way that the orientations of the fibers are identified is given by classifying homomorphism ϕ . Thus if G_0 is orientable, then $\pi_1(M_0)$ is generated by h , the a_i , b_i , c_i and d_i with relations $y_i h y_i^{-1} = h^{\phi(y_i)}$ where y is any of the letters a , b , c or d . Note that $\phi(c_i) = +1$ since each c_i bounds a disk when G_0 is viewed as a subset of G .

If G_0 is non-orientable, then $\pi_1(M_0)$ is generated by h , the x_i , c_i and d_i with the same relations.

Note that the realization of M_0 has a section of G_0 naturally embedded in it. The generators of $\pi_1(G_0)$ correspond to the like named generators of $\pi_1(M_0)$. In particular, the generators c_i , d_i and e correspond to curves on the boundary components of M_0 that are the boundaries of the section of G_0 . Since these curves are crossing curves, we have on each of the boundary components of M_0 , a pair of curves that generate the fundamental group of that boundary component. One is h and the other is one of c_i , d_i or e . Note that if the boundary components c_i and e of G_0 are filled in with disks, then the orbit surface of M' is obtained. We obtain M' from M_0 by sewing ordinary solid tori to the boundary components of M_0 over the c_i with meridians sewn to the c_i and an ordinary solid torus to the component over e with meridian sewn to $e + bh$. Note that this trivializes each c_i in the above presentations and trivializes the curve eh^b where e is as given in either 2.1 or 2.2.

To reconstruct M instead of M' , we have to sew fibered solid tori to the boundary components over the c_i , instead of ordinary solid tori. A meridian for the fibered solid torus sewn to the component over c_i is to be sewn to $h^{\beta_i} c_i^{\mu_i}$. We can now write out the presentation for $\pi_1(M)$.

THEOREM 2.2.2. *Let M be a compact, connected Seifert fibered space with m boundary components, with n exceptional fibers with crossing invariants $(\mu_1, \beta_1, \dots, \mu_n, \beta_n)$, with section obstruction b , and with classifying homomorphism ϕ . If the orbit surface of M is orientable of genus g , then $\pi_1(M)$ has the*

presentation

$$\begin{aligned}
 (2.3) \quad & \langle h, a_1, b_1, \dots, a_g, b_g, c_1, \dots, c_n, d_1, \dots, d_m \mid a_i h a_i^{-1} = h^{\phi(a_i)}, \\
 & b_i h b_i^{-1} = h^{\phi(b_i)}, \\
 & c_i h c_i^{-1} = h, \\
 & d_i h d_i^{-1} = h^{\phi(d_i)}, \\
 & c_i^{\mu_i} h^{\beta_i} = 1, \\
 & \prod [a_i, b_i] \prod c_i \prod d_i h^b = 1 \rangle.
 \end{aligned}$$

If the orbit surface of M is non-orientable with k crosscaps, then $\pi_1(M)$ has the presentation

$$\begin{aligned}
 (2.4) \quad & \langle h, x_1, \dots, x_k, c_1, \dots, c_n, d_1, \dots, d_m \mid x_i h x_i^{-1} = h^{\phi(x_i)}, \\
 & c_i h c_i^{-1} = h, \\
 & d_i h d_i^{-1} = h^{\phi(d_i)}, \\
 & c_i^{\mu_i} h^{\beta_i} = 1, \\
 & \prod x_i^2 \prod c_i \prod d_i h^b = 1 \rangle.
 \end{aligned}$$

■

Note that if ϕ is the trivial homomorphism, then h is a central element. If ϕ takes on the value -1 on some element, then h is not preserved under conjugation by that element, but the cyclic subgroup generated by h is preserved. Thus the cyclic subgroup generated by h is normal in $\pi_1(M)$. The element h is represented by some ordinary fiber in M . Another ordinary fiber in M is isotopic in M to the one representing h and so represents an element conjugate to h or h^{-1} . Since the cyclic subgroup generated by h is normal in $\pi_1(M)$, any other ordinary fiber must represent either h or h^{-1} and generate the same cyclic subgroup as that generated by h . This is summarized in the next lemma.

COROLLARY 2.2.2.1. *The fundamental group of a Seifert fiber space has a unique cyclic normal subgroup (possibly trivial) which is generated by any ordinary fiber.* ■

Note that this does not claim that there are no other cyclic normal subgroups in $\pi_1(M)$.

Using the notation of Theorem 2.2.2, we let $\langle h \rangle$ denote the cyclic, normal subgroup of $\pi_1(M)$ generated by an ordinary fiber. We can form the quotient group $\pi_1(M)/\langle h \rangle$. This has one of the following presentations depending on the orientability of the orbit surface of M :

$$(2.5) \quad \langle a_1, b_1, \dots, a_g, b_g, c_1, \dots, c_n, d_1, \dots, d_m \mid c_i^{\mu_i} = 1, \prod [a_i, b_i] \prod c_i \prod d_i = 1 \rangle,$$

or

$$(2.6) \quad \langle x_1, \dots, x_k, c_1, \dots, c_n, d_1, \dots, d_m \mid c_i^{\mu_i} = 1, \prod x_i^2 \prod c_i \prod d_i = 1 \rangle.$$

Groups with these presentations are known as Fuchsian groups about which a great deal is known. They appear in complex analysis. Later we will use these groups to show that some of the spaces excluded from Lemma 2.2.1 are P^2 -irreducible. We first give a lemma relating the size of the fundamental group of some Seifert fibered spaces to the size of the cyclic normal subgroup generated by an ordinary fiber and the size of the corresponding Fuchsian quotient.

LEMMA 2.2.3. *Let M be a compact, connected, P^2 -irreducible and boundary irreducible Seifert fibered space with infinite fundamental group. Let $h \in \pi_1(M)$ be represented by an ordinary fiber. Then the cyclic normal subgroup $\langle h \rangle$ generated by h is infinite and the Fuchsian group $\pi_1(M)/\langle h \rangle$ is infinite.*

PROOF Since M is P^2 -irreducible with infinite fundamental group it is aspherical. Since it is finite dimensional, its fundamental group is torsion free.

Thus h is trivial or has infinite order. If h is trivial, then $\pi_1(M)$ is a Fuchsian group which must then be torsion free and infinite. Also, ∂M must be empty by Lemma 2.1.4. From the presentation of a Fuchsian group, we know that a torsion free, infinite Fuchsian group is the fundamental group of an aspherical surface S . Thus M and S have the same homotopy type. Since M is closed, $H_3(M; \mathbf{Z}_2)$ is not zero and we have a contradiction.

If $\pi_1(M)/\langle h \rangle$ is finite, then a finite cover of M has infinite cyclic fundamental group. The cover will also be aspherical and boundary irreducible. However, this is impossible. (This is standard. There is a homotopy equivalence to a circle. Make the map transverse to a point and look at the preimage of the point. Make the preimage an incompressible surface. It can only be a non separating 2-sphere or disk violating either the irreducibility or the boundary irreducibility.) ■

COROLLARY 2.2.3.1. *Let M be a compact connected, Seifert fibered space with non-empty boundary that is not a fibered solid torus. Let $h \in \pi_1(M)$ be represented by an ordinary fiber. Then $\pi_1(M)$, the cyclic normal subgroup $\langle h \rangle$ generated by h , and the Fuchsian group $\pi_1(M)/\langle h \rangle$ are all infinite.*

PROOF This follows from Lemma 2.1.4, Lemma 2.1.5 and Lemma 2.2.3. ■

We now consider some very special cases of Fuchsian groups.

2.2.2. The dihedral groups and the group $\mathbf{Z}_2 * \mathbf{Z}_2$. We will be interested in knowing which spaces are not P^2 -irreducible. A first step will be to determine the spaces that are not irreducible. The fundamental groups of such spaces will be free products. (If there are homotopy spheres that are not S^3 , then the free product may be trivial.) We will show that $\mathbf{Z}_2 * \mathbf{Z}_2$ is the only non-trivial free product that has a non-trivial cyclic normal subgroup. This is the “infinite dihedral” group, and it and the finite dihedral groups turn out to be relevant. [Actually, the finite ones may not be all that relevant. Read on.]

The dihedral group D_n of order $2n$ has presentation

$$\langle x, t \mid x^2 = t^n = 1, txt = t^{-1} \rangle$$

or

$$\langle x, t \mid x^2 = t^n = txt = 1 \rangle$$

or

$$(2.7) \quad \langle x, t \mid x^2 = t^n = (xt)^2 = 1 \rangle.$$

Letting $y = xt$ so that $t = xy$ we get the alternate presentation

$$(2.8) \quad \langle x, y \mid x^2 = y^2 = (xy)^n = 1 \rangle.$$

The infinite dihedral group D_∞ eliminates the cyclicity in the generator t and has presentations

$$(2.9) \quad \langle x, t \mid x^2 = (xt)^2 = 1 \rangle$$

and

$$(2.10) \quad \langle x, y \mid x^2 = y^2 = 1 \rangle$$

using the same substitutions. However the last is just a presentation of $\mathbf{Z}_2 * \mathbf{Z}_2$.

We stretch notation and use D_n to refer to both the finite and infinite dihedral groups. We will specify n finite when needed and use D_∞ when needed.

Note that D_n for finite n is isomorphic to the Fuchsian group

$$\langle c_1, c_2, c_3 \mid c_1^2 = c_2^2 = c_3^n = c_1c_2c_3 = 1 \rangle$$

since $c_3 = (c_1c_2)^{-1}$ and $(c_1c_2)^{-n} = 1$ if and only if $(c_1c_2)^n = 1$. Similarly D_∞ is isomorphic to the Fuchsian group

$$\langle c_1, c_2, d_1 \mid c_1^2 = c_2^2 = c_1c_2d_1 = 1 \rangle.$$

The subgroup $\langle t \rangle$ of D_n generated by t is cyclic and normal as are all of the subgroups of $\langle t \rangle$. We investigate the other normal subgroups.

It is clear from the presentations 2.8 and 2.10 that elements of D_n are words that alternate in the letters x and y . A word with an even number of letters is a power (positive or negative) of $t = xy$, and a word with an odd number of letters is a conjugate of the letter in the center of the word, either x or y . Thus words with an odd number of letters are involutions. The elements x and y are not conjugate since setting $x = 1$ in either 2.8 or 2.10 does not kill the element y . Note that setting $x = 1$ gives a presentation for \mathbf{Z}_2 , so the subgroup of D_n normally generated by x is of index 2 in D_n . Similarly for y . Since $\langle t \rangle$ is normal in D_n it is a union of conjugacy classes. Thus $D_n - \langle t \rangle$ holds two conjugacy classes, the conjugates of x and the conjugates of y . (So $D_n - \langle t \rangle$ consists entirely of involutions.) Thus a normal subgroup of D_n either is a subgroup of $\langle t \rangle$, or contains the conjugates of x or y or both. If a normal subgroup contains conjugates of both x and y , then it is all of D_n . If it contains conjugates of x , then it contains x and xyx and thus $xyxy = t^2$. Thus the index 2 normal subgroup containing x consists of words in x and y that either have odd length and center letter x or have length a multiple of 4. Similarly for y . Neither of these normal subgroups is cyclic in D_n since neither x nor y commute with $t^2 = xyxy$. Thus the cyclic normal subgroups of D_n are precisely the subgroups of $\langle t \rangle$. Note that modding out by one of these cyclic normal subgroups means adding the relation $t^d = 1$ to 2.7 for some divisor d of n or to 2.9 for any d . The quotient is the dihedral group

$$\langle x, t \mid x^2 = t^d = (xt)^2 = 1 \rangle \cong \langle x, y \mid x^2 = y^2 = (xy)^d = 1 \rangle$$

if $d \neq 1$ or \mathbf{Z}_2 if $d = 1$.

We have the following uniqueness result.

LEMMA 2.2.4. *The only non-trivial free product with a non-trivial cyclic normal subgroup is D_∞ .*

PROOF Assume that $G = A * B$ is a non-trivial free product and that G has a non-trivial cyclic normal subgroup N . Every element of G is of the form $a_1 b_1 \dots a_n b_n$ where each a_i is in A and each b_i is in B . No $a_i = 1$ except possibly a_1 and no $b_i = 1$ except possible b_n . Such an element is trivial if and only if $n = 1$ and $a_1 = b_1 = 1$. Assume such an element is the generator of N .

If N is finite cyclic, then it is conjugate into one of A or B . Since it is normal, it would be in one of A or B . However, conjugating an element of A by an element of B produces an element that is not in A . Thus N is not finite.

Since N is normal all rotations of the generator are in N and are non-trivial. Thus by passing to a subgroup of N , we may assume that the generator g of N is cyclically reduced and has both $a_1 \neq 1$ and $b_n \neq 1$. Now positive powers of g start with a_1 and negative powers start with b_n^{-1} . The length (measured in numbers of letters) of g^k is $2n|k|$. Conjugating g by $a_1 b_1$ produces $a_2 b_2 \dots a_n b_n a_1 a_2$ which can only equal g . Thus each $a_i = a_1$ and each $b_i = b_1$. Thus $g = (a_1 b_1)^n$ and every element of N is of the form $(a_1 b_1)^{nk}$. Conjugating g by a_1 produces $(b_1 a_1)^n$ which can only equal g^{-1} so $a_1 = a_1^{-1}$ and $b_1 = b_1^{-1}$. Assume that A has an element c other than a_1 . We know $c^{-1} \neq a_1$ as well. Conjugating g by c produces $a'(b_1 a_1)^{n-1} b_1 c$ where $a' = c^{-1} a_1 \neq 1$. This is no element of N . Thus a_1 is the only element of A . Similarly, b_1 is the only element of B . Thus G is $\mathbf{Z}_2 * \mathbf{Z}_2$ or D_∞ . ■

Next we take up a larger class of Fuchsian groups.

2.2.3. The triangle groups. Let p, q and r be integers each greater than

1. The Fuchsian group

$$\langle c_1, c_2, c_3 \mid c_1^p = c_2^q = c_3^r = c_1 c_2 c_3 = 1 \rangle$$

is isomorphic to

$$\langle c_1, c_2 \mid c_1^p = c_2^q = (c_1 c_2)^r = 1 \rangle.$$

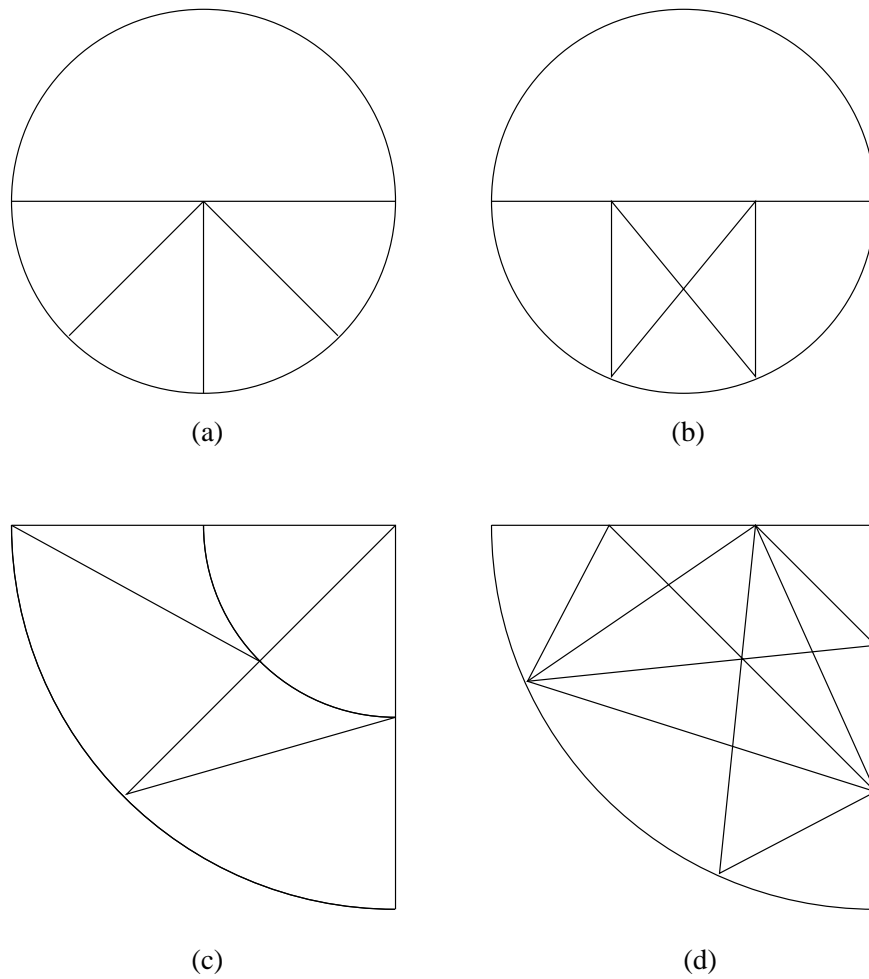
The presentations make it clear that permutations of the subscripts $\{123\}$ give isomorphic groups so we may assume that $p \leq q \leq r$. These groups are called "triangle groups". The reason for this terminology will become clear. (Actually, these groups should probably be called "hexagonal groups." Another set of groups has better claim to the title of triangle groups. However, the notation seems to be standard.)

Let $\Gamma(p, q, r)$ have the presentation $\langle a, b, c \mid a^p = b^q = c^r = abc = 1 \rangle$ or equivalently $\langle a, b \mid a^p = b^q = (ab)^r = 1 \rangle$. We will use Γ for short when the specific values of (p, q, r) are not important or are clear from the context. Let Δ be a triangle with vertex angles π/p , π/q and π/r . Such a triangle can be realized with geodesic sides on the sphere S^2 if the sum of the angles is greater than π , on the Euclidean plane E^2 if the sum of the angles is exactly π and on the hyperbolic plane H^2 if the sum of the angles is less than π . This is determined by comparing the sum

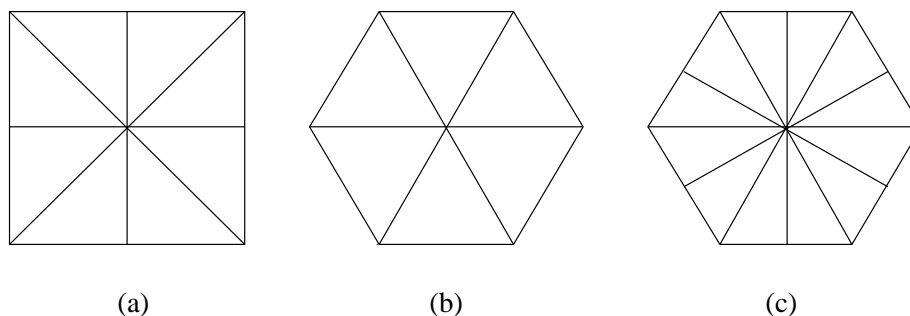
$$(2.11) \quad \frac{1}{p} + \frac{1}{q} + \frac{1}{r}$$

to 1. The triangle is realized on S^2 , E^2 or H^2 respectively as 2.11 is greater than, equal to or less than 1. Since each of p , q and r is at least 2, we can easily describe the triples (p, q, r) , up to permutation, for which 2.11 is no less than 1. The sum is greater than 1 for $(2, 2, r)$, $(2, 3, 3)$, $(2, 3, 4)$ and $(2, 3, 5)$. The sum is exactly 1 for $(2, 3, 6)$, $(2, 4, 4)$ and $(3, 3, 3)$.

We will show that if we reflect Δ across its edges repeatedly, we will tessellate S^2 , E^2 or H^2 with copies of Δ . Figure 2.1 shows how to tessellate S^2 for the values $(2, 2, 4)$, $(2, 3, 3)$, $(2, 3, 4)$ and $(2, 3, 5)$ of (p, q, r) . In Figure 2.1, (a) and (b) show a quadrant of S^2 as viewed from above, while (c) and (d) show an octant of S^2 as viewed from above. Figure 2.2 shows basic units of tessellations

FIGURE 2.1. Tessellations of S^2

of E^2 for the allowable values of (p, q, r) . For examples of tessellations of H^2 see various etchings of M. C. Escher.

FIGURE 2.2. Tessellations of E^2

The original argument that I had for what follows was flawed. The argument below is mostly derived from B. Maskit, “On Poincaré’s theorem for fundamental polygons,” *Advances in Math.*, **7** (1971), 219–230.

That the triangles successfully tessellate their respective spaces is a covering argument. When we do this we will be analyzing a group that contains Γ as a subgroup of index 2. (This larger group is one that more naturally claims the title of a triangle group.) We could do the argument directly with Γ (using hexagons), but it is easier to work with the triangles, the larger groups is interesting in its own right, the use of triangles shows why Γ is called a triangle group, and once the argument for the larger group is done, the argument for Γ is seen as virtually identical.

We actually use two isomorphic groups in the argument. At first it is not clear that the groups are isomorphic. One is defined geometrically as a group of reflections, and the other by a presentation. We will get our tessellation by showing that an abstract complex defined from the presentation is a covering space for the space acted on by the geometric group. We get as a further consequence that the two groups are isomorphic. This will be used later to relate Γ to a subgroup of the geometric group.

We start with the group of reflections. Let S be whichever of S^2 , E^2 or H^2 contains Δ . We start a pattern of abusing notation and use x , y and z to label the edges of the triangle Δ . We use x to label the edge opposite the vertex of angle π/q , we use y to label the edge opposite the vertex of angle π/r , and we use z to label the edge opposite the vertex of angle π/p . We let G be the group of isometries of S generated by reflections in the sides of Δ . That is, reflections in the lines that contain the sides of Δ . Continuing our abuse of notation, we let x , y and z denote reflection of S across the lines of S that contain the edges x , y and z of Δ respectively. Thus G is generated by the elements x , y and z . We let elements of G act on the left. Since elements of G are isometries, they are determined by their actions on Δ . If g is an element of G , then we let the triangle $g(\Delta)$ have its edges labeled by x , y and z as carried over by the action of g . (This is not so much a labeling of a subset of S , but a labeling of an image of a specific map. We are not going to be concerned with the fact that two elements of G might have the same images when restricted to Δ .)

Note that each of x , y and z is an involution in G and that each is non-trivial. Also, xy is a rotation about the common point of the lines of reflection for x and y . That is, xy is a rotation about the vertex of Δ that is common to the sides x and y . This is the vertex opposite the side z , and in our notation it is the vertex of angle π/p . The element xy of G is seen to be a rotation about this vertex by an angle $2\pi/p$. Thus xy has order exactly p in G . Similarly, yz has order q and is a rotation by $2\pi/q$ about the vertex of Δ of angle π/q , and zx has order r and is a rotation by $2\pi/r$ about the vertex of Δ of angle π/r . We use the relations that we have discovered about x , y and z in G to define an abstract group.

Let G_0 be the group with presentation

$$\langle x, y, z \mid x^2 = y^2 = z^2 = (xy)^p = (yz)^q = (zx)^r = 1 \rangle.$$

The group G is a homomorphic image of G_0 by a homomorphism taking the generators of G_0 to the elements of the same letter in G . We will see that the

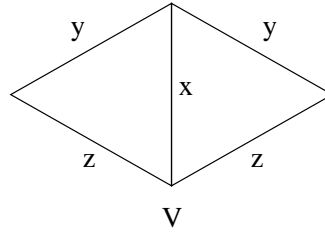


FIGURE 2.3. Identified triangles

homomorphism just defined is actually an isomorphism between G_0 and G . From what we know of G , we know that each of x , y and z is a non-trivial involution in G_0 and that xy , yz , and zx have orders p , q and r respectively in G_0 .

Let Ω be the graph of the group G_0 with respect to the generating set x , y and z . Recall that the edges of the graph Ω are also labeled with the letters x , y and z . Also, recall that the edges of Δ are labeled with the letters x , y and z . We use the graph Ω to define a simplicial complex using copies of the triangle Δ .

Let $K = G_0 \times \Delta$. Let g and gw be elements in G_0 where w is one of x , y or z . Thus g and gw are connected by an edge in Ω . (Remember that each of x , y and z is its own inverse.) In K , we will now identify the edges labeled w in the triangles (g, Δ) and (gw, Δ) . Since Δ is a geometric object, we can insist that the identification is done by an isometry. The only thing that we have to specify is the orientation of the sewing. We require that the sewing be done so that the triangles (g, Δ) and (gw, Δ) are seen as reflections of each other across the common edge w . Another way of specifying this is to require that for each endpoint of w , the edges of (g, Δ) and (gw, Δ) other than w that touch that endpoint (after identification) have the same label. See Figure 2.3 for an example where $w = x$. We let F represent the resulting complex. Since the graph of G_0 is connected, we get that the complex F is connected.

Since every edge of a triangle in K is labeled with one of x , y and z , and since each of x , y and z is a non-trivial involution in G_0 , we know that every edge of a triangle in K is identified with exactly one other edge of a triangle in K . Thus F is a surface except possibly at the images of the vertices. However, each of xy , yz and zx have finite order in G_0 . This implies that each vertex has a surface neighborhood. (Consider the implications of the fact that zx has finite order on the vertex labeled V in Figure 2.3.) Thus F is a surface without boundary.

If g is an element of G , and w is one of x , y or z , then the triangles $g(\Delta)$ and $gw(\Delta)$ are the images under g of Δ and $w(\Delta)$ and are thus reflections of each other in the edge w . If we now map (g, Δ) in K to $g(\Delta)$ and (gw, Δ) in K to $gw(\Delta)$ by isometries that preserve the labeled edges, then these maps commute with the identifications that create F from K .

We now map each (g, Δ) in K to $g(\Delta)$ in S by an isometry that preserves the labeled edges. This gives a map from F to S that is an isometry on each triangle. Further, this map is a local isometry at each point in the interior of an edge since the triangles that meet along an edge in F are carried to triangles that are reflections of each other in S . Also, the map is a local isometry at each vertex, since the number of triangles meeting at a vertex of F is carried to the same number of triangles meeting at a vertex in S . We thus have that the map is a local isometry on the star of each vertex.

We now have a local homeomorphism $f : F \rightarrow S$ and we wish to show that it is a covering projection. We must show that it evenly covers. In the surface F , there is an ϵ so that every ball of radius ϵ is contained in the star of some vertex. If f is not onto and a is a limit point of the image, then there is a point b in F with $f(b)$ less than ϵ from a . But the ϵ ball about b maps isometrically to S and will contain a point mapping to a . Thus the image of f is closed. Since f is a local homeomorphism, its image is also open and must be all of S . Now

if a is any point in S , then the ϵ ball about a is evenly covered by f . Thus f is a covering projection.

The space S is simply connected and the surface F is connected. Thus f must be a homeomorphism and S is tessellated by the images of Δ under the action of G .

We know that the homomorphism from G_0 to G is an epimorphism since x , y and z generate G . We see that this homomorphism is an isomorphism by noting that different elements g and g' of G_0 correspond to different triangles (g, Δ) and (g', Δ) in F which map to different triangles $g(\Delta)$ and $g'(\Delta)$ of S (since h is a homeomorphism) so that g and g' must be different elements of G . We now drop the notation G_0 and use only G from now on.

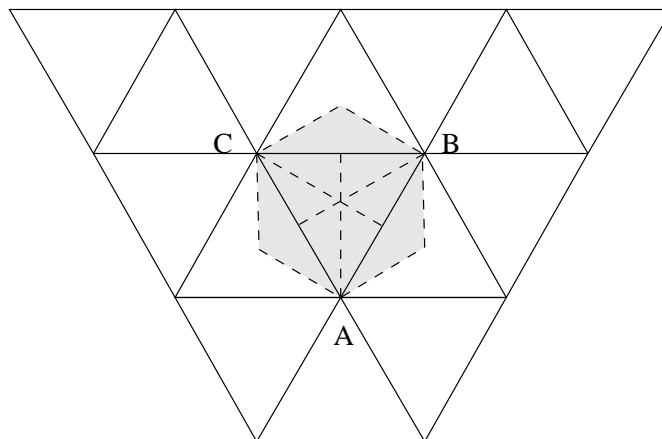
We are now ready to consider Γ .

The subgroup G_1 of G generated by $A = xy$, $B = yz$ and $C = zx$ satisfies the relations $A^p = B^q = C^r = ABC = 1$ and so is a homomorphic image of Γ by sending a to A , b to B and c to C . We must show that this gives an isomorphism from Γ to G_1 .

The homomorphism already shows that a , b and c have exactly the orders p , q and r respectively in Γ because of the presentation of Γ and because these are exactly the orders in G_1 .

Note that the action of A is to rotate by $2\pi/p$ around the vertex shared by the edges x and y . We again abuse notation and use A to represent the vertex shared by the edges x and y . Similarly, B is the vertex shared by y and z , and C is the vertex shared by z and x . The action of B in G_1 is to rotate by $2\pi/q$ about the vertex B , and the action of C in G_1 is to rotate by $2\pi/r$ about the vertex C .

Let Δ and S be as above with the edges of Δ labeled x , y and z as before. For w one of x , y or z , let Δ_w be the union of the two triangles in the barycentric

FIGURE 2.4. The Hexagon Ξ

subdivision of Δ that share an edge with w . We have that Δ is the union of the three triangles Δ_x , Δ_y and Δ_z . The three triangles have disjoint interiors, and each Δ_w is a neighborhood in Δ of the edge w . Recall that w is also a reflection. Thus $w(\Delta_w)$ is a triangle in $w(\Delta)$ that is a neighborhood of w in $w(\Delta)$ and $\Delta_w \cap w(\Delta_w)$ is a neighborhood of w in S .

We now let Ξ be the hexagon

$$\Delta \cap x(\Delta_x) \cap y(\Delta_y) \cap z(\Delta_z).$$

The vertices A , B and C are now three of the six vertices of Ξ . See Figure 2.4. The angle in Ξ at each of these vertices is twice the angle found in Δ . Thus the angle in Ξ is $2\pi/p$ at A , $2\pi/q$ at B and $2\pi/r$ at C .

It is now possible to imitate the proof that Δ tessellates S under the action of G to show that Ξ tessellates S under the action of G_1 . One forms a surface from $\Gamma \times \Xi$ and shows that it is a cover of S . Slightly more care is needed to define the identifications since the directions of the rotations need to be taken into account.

The important fact is that the action of G_1 produces surface neighborhoods of the edges of Ξ and the vertices A , B and C . We leave the details to the reader. It also follows in exactly the same way that Γ and G_1 are isomorphic. We now drop the notation G_1 and use only Γ from now on.

We have now identified $\Gamma(p, q, r)$ with a certain group of symmetries of a triangular (actually hexagonal) tessellation of S^2 , E^2 or H^2 . The group of symmetries is nice in that it sets up a one to one correspondence between the elements of $\Gamma(p, q, r)$ and some of the triangles in the tessellation. Namely, we can associate the element g in Γ with the triangle $g(\Delta)$ in S . The triangles used in this way are “half” the triangles in S (since the fundamental domain Ξ for Γ is twice the size of the fundamental domain Δ for G) and are the triangles in S that are obtained from Δ by orientation preserving elements of G . Thus Γ is the index two subgroup of G consisting of all the orientation preserving elements of G .

Note that the vertices of the triangular tessellation of S can be given well defined labels from the alphabet $\{ABC\}$. This is because each triangle in the tessellation is the image of Δ under a unique element of G , and because when two triangles $g(\Delta)$ and $g'(\Delta)$ in the tessellation share an edge $g(w)$, then $g|w = g'|w$. Thus for a vertex V in the tessellation, we give it the same label as a vertex of Δ that is carried to V by any element of G .

We can compute the orders of the triangle groups that are finite. These are the groups $\Gamma(2, 2, r)$, $\Gamma(2, 3, 3)$, $\Gamma(2, 3, 4)$ and $\Gamma(2, 3, 5)$. We will do an Euler characteristic argument. Let V , E and F be the number of vertices, edges and faces of the triangular tessellation. The vertices break into three classes, one label A and angles π/p , one with label B and angles π/q and one with label C and angles π/r . We use the letters P , Q and R to represent the number of vertices in the respective classes. Thus $V = P + Q + R$. Each face is a triangle

with exactly one vertex of each class. Thus $2pP = 2qQ = 2rR = F$. Also $3F = 2E$. So

$$\chi(S^2) = 2 = V - E + f = \frac{F}{2} \left(\frac{1}{p} + \frac{1}{q} + \frac{1}{r} - 1 \right).$$

This allows us to get F for the various values of (p, q, r) . The order of Γ is one half F so we get $|\Gamma(2, 2, r)| = 2r$, $|\Gamma(2, 3, 3)| = 12$, $|\Gamma(2, 3, 4)| = 24$ and $|\Gamma(2, 3, 5)| = 60$. The first is not surprising since $\Gamma(2, 2, r)$ is the dihedral group of order $2r$.

The next paragraph is not needed. I did not realize this when I put it in, and I don't want to take it out.

We would like to argue that the finite groups are all different. The only problem might be that $\Gamma(2, 3, 3)$, $\Gamma(2, 3, 4)$ or $\Gamma(2, 3, 5)$ might be dihedral groups. Taking the presentations as $\langle a, b \mid a^2 = b^q = (ab)^r = 1 \rangle$, we can compute the abelianizations. For $\Gamma(2, 2, r)$ we get $2a = 0$, $2b = 0$ and $ra + rb = 0$. If r is even, we get no new information from $ra + rb = 0$ and the abelianization is $\mathbf{Z}_2 \oplus \mathbf{Z}_2$. If r is odd, then $ra + rb = 0$ reduces to $a + b = 0$ so $a = -b$ and the abelianization is \mathbf{Z}_2 . Similar calculations show that $\Gamma(2, 3, 3)$ abelianizes to \mathbf{Z}_3 , $\Gamma(2, 3, 4)$ abelianizes to \mathbf{Z}_2 and $\Gamma(2, 3, 5)$ abelianizes to the trivial group. The only possibility for a dihedral group is $\Gamma(2, 3, 4)$ but its order is a multiple of 4 and the dihedral group of that order abelianizes to $\mathbf{Z}_2 \oplus \mathbf{Z}_2$. Thus the finite triangle groups are all different.

We return to an arbitrary triangle group $\Gamma(p, q, r)$. Drawing pictures shows that the rotations λ and κ do not commute except when $p = q = r = 2$. In this case there are four triangles being permuted on S^2 by a group of order four with three involutions. Thus Γ is not cyclic.

If Γ is a non-trivial free product, then either Γ is $\mathbf{Z}_2 * \mathbf{Z}_2$ and Γ is two ended, or one of the free factors has order at least 3 and Γ is infinitely ended. However, Γ acts as a group of covering transformations on a simply connected space with

compact quotient. The simply connected space is either homeomorphic to S^2 or R^2 and so Γ can have only zero ends or one end. In no case can Γ be a non-trivial free product.

2.3. Properties of “small” of Seifert fibered spaces

We start work on the excluded spaces in Lemma 2.2.1. We will break the discussion into cases.

2.3.1. Orbit surface S^2 and less than four exceptional fibers. All Seifert fibered spaces with orbit surface S^2 are orientable and cannot contain two sided projective planes. Thus showing irreducibility is sufficient to show P^2 -irreducibility. Let M be a closed Seifert fibered space with orbit surface S^2 and exactly three exceptional fibers.

LEMMA 2.3.1. *The Heegaard genus of M is no more than 3.*

PROOF Let A be the union of pairwise disjoint fibered solid torus neighborhoods of the exceptional fibers, and let B be the closure of the complement of A . We know that the classifying homomorphism of M is trivial, so B is the product of a disk with two holes with S^1 . That is B is the double along an annulus of a product of an annulus with S^1 . A hole can be drilled in an annulus cross S^1 to connect the two boundary components in such a way that a handlebody with of genus two results. The hole can miss a given annulus in the boundary. The annulus will then become a π_1 generator for one of the handles. Thus we can drill two holes in B to give the double of two handlebodies of genus 2 doubled along this annulus. The result is a handlebody of genus 3. The holes drilled in B are added to A as 1-handles in such a way as to connect the boundary components of A . This connects A and gives another handlebody of genus 3. ■

We are interested in whether M is irreducible. Assume that it is not. Then it is a connected sum of two 3-manifolds, one of which is genus 1, and $\pi_1(M)$ is a non trivial free product, or cyclic. (Actually the only simply connected 3-manifold of Heegaard genus 2 is S^3 , but this is hard to prove and we don't need it.)

A presentation for $\pi_1(M)$ is

$$\begin{aligned} \langle h, c_1, c_2, c_3 \mid c_1 h c_1^{-1} = c_2 h c_2^{-1} = c_3 h c_3^{-1} = h, \\ c_1^{\mu_1} h^{\beta_1} = c_2^{\mu_2} h^{\beta_2} = c_3^{\mu_3} h^{\beta_3} = 1, \\ c_1 c_2 c_3 h^b = 1 \rangle. \end{aligned}$$

Each μ_i is at least 2. Since $(\mu_i, \beta_i) = 1$, each β_i is not zero. If we mod out by the normal cyclic subgroup $\langle h \rangle$, then we obtain a triangle group $\Gamma(\mu_1, \mu_2, \mu_3)$. This is not cyclic, so $\pi_1(M)$ is not cyclic. If $\langle h \rangle$ is trivial, then $\pi_1(M)$ is isomorphic to $\Gamma(\mu_1, \mu_2, \mu_3)$. But $\Gamma(\mu_1, \mu_2, \mu_3)$ is neither cyclic nor a non-trivial free product. Thus $\langle h \rangle$ is non-trivial, $\pi_1(M)$ is not cyclic, and $\pi_1(M)$ must be a non-trivial free product.

Since the only non-trivial free product with a non-trivial cyclic, normal subgroup is $\mathbf{Z}_2 * \mathbf{Z}_2$, we have $\pi_1(M) \cong \mathbf{Z}_2 * \mathbf{Z}_2$. We know that $\langle h \rangle$ is contained in the unique maximal cyclic, normal subgroup of $\mathbf{Z}_2 * \mathbf{Z}_2 \cong D_\infty$ and that h has infinite order. Each element not in the maximal normal, cyclic subgroup of $\mathbf{Z}_2 * \mathbf{Z}_2$ is an involution. However, each generator c_i has a power equal to a non-zero power of h and so each c_i has infinite order and is not an involution. Thus all of the generators are in the maximal cyclic, normal subgroup and there can be no involutions. Since an irreducible, orientable 3-manifold is P^2 -irreducible, we have shown:

LEMMA 2.3.2. *A closed Seifert fibered space with orbit surface S^2 and exactly three exceptional fibers is P^2 -irreducible.* ■

Note that we know which of these spaces have infinite fundamental group.

LEMMA 2.3.3. *Let M be a closed Seifert fibered space with orbit surface S^2 and exactly three exceptional fibers. Let $h \in \pi_1(M)$ be represented by an ordinary fiber. If the indexes of the exceptional fibers are not one of $(2, 2, r)$, $(2, 3, 3)$, $(2, 3, 4)$ or $(2, 3, 5)$, then $\pi_1(M)$, $\langle h \rangle$ and $\pi_1(M)/\langle h \rangle$ are all infinite. Otherwise all three groups are finite.*

PROOF We know that M is P^2 -irreducible and boundary irreducible. The result follows immediately from Lemma 2.2.3 and the fact that the only finite triangle groups are the ones specified by the triples listed in the hypothesis. ■

We can also give a criterion for having an incompressible surface.

LEMMA 2.3.4. *Let M be a closed Seifert fibered space with orbit surface S^2 and exactly three exceptional fibers. Then M has an incompressible surface if and only if $H_1(M)$ is infinite.*

PROOF M has a non-separating incompressible surface if and only if $H_1(M)$ is infinite. Thus we must show that M has no separating incompressible surface when $H_1(M)$ is finite. What is actually true is that no closed Seifert fibered space with orbit surface S^2 and exactly three exceptional fibers has a separating incompressible surface.

Assume that M has a separating incompressible surface. Let N_1 , N_2 and N_3 be pairwise disjoint fibered solid torus neighborhoods of the exceptional fibers. Let S be a separating incompressible surface in M whose intersections with the N_i are meridional disks of the N_i and so that the total number of these disks is minimal. Let M' be M with the interiors of the N_i removed and let S' be $S \cap M'$. If S' compresses in M' along a non-separating curve in S' , then S is not incompressible in M . If S' compresses in M' along a separating curve J in

S' , then J bounds a disk in S which can be replaced by the compressing disk to lower the number of boundary components of S' . Thus the minimality of the number of boundary components of S' implies the incompressibility of S' in M' .

The space M' is the product of S^1 with a disk with two holes E and $\pi_1(M')$ has non-trivial center. Also, $\pi_1(M')$ is a free product amalgamated over $\pi_1(S')$ so either $\pi_1(S')$ is abelian, or S' carries the fundamental group of one of its complementary domains. In the latter case, S' would be boundary parallel and would be an annulus or a torus. In the former case, S' must be a sphere, annulus or a torus. Since M and M' are irreducible, a sphere is ruled out. If S' is an annulus, then S is a sphere and this case is also ruled out.

If S' is a separating incompressible torus in M' , it would hit E in simple closed curves. We can eliminate the curves that are trivial on either E or S' . Elements of $\pi_1(E)$ that are conjugate in $\pi_1(M')$ are conjugate in $\pi_1(E)$, so the curves must all be parallel. Since E is planar, the curves all separate E . Since E has only three boundary components, the curves must be parallel to one of them. However, these curves demonstrate that $S' = S$ is compressible in M . Thus S' is disjoint from E . This puts S' in a product of a disk with three holes with an interval and puts $\mathbf{Z} \oplus \mathbf{Z}$ as a subgroup of the free group on two generators. This gives a contradiction. ■

REMARK From the presentation for $\pi_1(M)$, we get that $H_1(M)$ is an abelian group with four generators with relation matrix

$$\begin{pmatrix} \beta_1 & \mu_1 & 0 & 0 \\ \beta_2 & 0 & \mu_2 & 0 \\ \beta_3 & 0 & 0 & \mu_3 \\ b & 1 & 1 & 1 \end{pmatrix}.$$

If this matrix is of full rank over \mathbf{Z} , then $H_1(M)$ is finite. It is easy create triples (μ_1, μ_2, μ_3) that make $\pi_1(M)$ infinite and chose the β_i and b so that $H_1(M)$

is finite. These were the first examples discovered of closed, P^2 -irreducible 3-manifolds with infinite π_1 and no incompressible surfaces. There are now many more known.

We now consider a Seifert fibered space M with orbit surface S^2 and with fewer than three exceptional fibers.

LEMMA 2.3.5. *The Heegaard genus of M is no more than 1.*

PROOF There is a circle in the orbit surface whose complementary domains are disks containing no more than one exceptional point. The preimages of these disks are fibered solid tori in M and represent M as a closed 3-manifold with Heegaard genus no more than 1. ■

Thus M is a lens space. If M is not irreducible, then Haken's theorem gives us a sphere that intersects each solid torus in the Heegaard splitting in a disk. This shows that M is $S^2 \times S^1$. We know that $S^2 \times S^1$ is in fact realizable as a Seifert fibered space with orbit surface S^2 and with zero or two exceptional fibers. We have shown:

LEMMA 2.3.6. *The only closed Seifert fibered space that is not P^2 -irreducible and that has an orientable orbit surface is $S^2 \times S^1$.* ■

We can also analyze the other properties that we have been considering.

LEMMA 2.3.7. *Let M be a closed, Seifert fiber space with orbit surface S^2 and with no more than two exceptional fibers. Then $\pi_1(M)$ is cyclic, generated by some fiber (possibly exceptional) and is finite except when M is $S^2 \times S^1$. In all cases, the associated Fuchsian group is finite. The only case in which M has an incompressible surface is when M is $S^2 \times S^1$ in which case M has a 2-sphere that bounds no 3-cell.*

PROOF This all follows from the structure of lens spaces. The generator of the fundamental group of a lens space is a centerline of one of the solid tori in a Heegaard splitting. An ordinary fiber is a power of this centerline. An incompressible surface in a closed, orientable 3-manifold with finite fundamental group must be a separating sphere. The only reducible lens space is $S^2 \times S^1$. ■

2.3.2. Orbit surface P^2 and less than two exceptional fibers. If M has orbit surface P^2 , then there are only two possible classifying homomorphisms. The trivial homomorphism yields a non-orientable M , and the non-trivial homomorphism yields an orientable M . As in the case of orbit surface S^2 , we only have to show that M is irreducible when M is orientable. This will be easy because all of the orientable spaces that we will be presented with except one will turn out to be spaces that we have analyzed before. The remaining space will turn out to be reducible. We will also discover that there are only two non-orientable spaces that arise. These will turn out not to be P^2 -irreducible. We will consider the cases of orientable M and non-orientable M separately.

Let M be a Seifert fibered space with orbit surface P^2 . If M has no more than 1 exceptional fiber, then there is a disk on P^2 whose preimage in M is a fibered solid torus T , and whose closed complement on P^2 is a Möbius band A that has no exceptional points. The preimage B of A in M is a circle bundle over A . The preimage S of the centerline of A is either a torus or a Klein bottle depending on the classifying homomorphism of M . If S is a torus, the S^1 -bundle over A is not twisted and M and B are non-orientable. If S is a Klein bottle, then the S^1 -bundle over A is twisted and M and B are orientable.

We can view B as an I -bundle over S and A is a subbundle over a circle in S . Since A is a Möbius band, the I -bundle is twisted. Thus the corresponding I -bundle over S must be twisted. There is only one class of classifying homomorphisms over a torus that involve twisting. There are two that involve twisting

over a Klein bottle, but only one will apply here because when S is a Klein bottle, B must be orientable.

Assume that S is a torus. Then there is no twisting along the centerline of A and B is just the product of A with a circle. This is the case where M is non-orientable and the classifying homomorphism for M is trivial. However we concentrate on the fact that B is an I -bundle over S . We let ϕ be the classifying homomorphism for B . We will have no reason to refer to the classifying homomorphism of the Seifert fibered space M during the rest of this case. Since there is only one class of classifying homomorphism over a torus that involves twisting, we can choose one representative for our convenience. Let (J, K) be a pair of generating curves for $H_1(S)$ that intersect in a point and assume that $\phi(J) = +1$ and $\phi(K) = -1$. Any self homeomorphism of S that preserves ϕ extends to a fiber preserving, self homeomorphism of B . (In fact there are two extensions. There is a homeomorphism of B of period two that fixes S and reverses each of the I -fibers. This can be composed with any extension.) Since the value of ϕ on $aJ + bK$ depends only on the parity of b , we know that a homeomorphism whose action on $H_1(S)$ is given by $\begin{pmatrix} a & c \\ b & d \end{pmatrix}$ must have b even and d odd. Since $\begin{vmatrix} a & c \\ b & d \end{vmatrix} = \pm 1$ it suffices to require that b be even.

The boundary of B is a torus S' that double covers S by pulling in along the fibers. The fibers over J form an annulus. One boundary of this annulus is a curve J' on S . The fibers over K form a Möbius band. The boundary of this Möbius band is a curve K' that represents $2K$ in the homology of B . The curves J' and K' intersect in a point and give a generating pair for $H_1(S')$.

The action of the homeomorphism of B that reverses all the I -fibers preserves the classes of J' and K' . Thus an allowable homeomorphism of S induces a well defined automorphism on $H_1(S')$ by extending it to one of the two fiber preserving, self homeomorphisms of B . We identify the homeomorphisms of S with the matrices that give the effect on $H_1(S)$.

Let a homeomorphism of S that preserves ϕ be given by $\begin{pmatrix} a & c \\ 2b & d \end{pmatrix}$. This homeomorphism takes J and the annulus over it to $\begin{pmatrix} a \\ 2b \end{pmatrix}$ and the annulus over it. One boundary component of this annulus runs a times around J and $2b$ times around K . This has it running a times in the J' direction on S' and only b times around the K' direction on S' . This can also be done with intersection numbers since only half of the intersections of the curve with the annulus over J are intersections with J' while all of the intersections with the Möbius band over K are intersections with K' . Thus the image of J' is $aJ' + bK'$ in $H_1(S')$.

The same homeomorphism takes K and the Möbius band over it to $\begin{pmatrix} c \\ d \end{pmatrix}$ and the Möbius band over it. The boundary of this Möbius band on S' runs twice over the curve $\begin{pmatrix} c \\ d \end{pmatrix}$ on S and an analysis similar to the image of J' shows that this curve is $2cJ' + dK'$ in $H_1(S')$. Thus the automorphism of $H_1(S')$ is given by $\begin{pmatrix} a & 2c \\ b & d \end{pmatrix}$. Any matrix $\begin{pmatrix} a & 2c \\ b & d \end{pmatrix}$ acting on $H_1(S')$ is realized by an extension of the homeomorphism of S that realizes $\begin{pmatrix} a & c \\ 2b & d \end{pmatrix}$ on $H_1(S)$.

Let $mJ' + nK'$ be a simple closed curve on S' . We know $(m, n) = 1$. If m is odd, then its prime factorization does not use the prime 2 and $(m, 2n) = 1$. There are integers a and c so that $am + 2cn = 1$. Thus $\begin{vmatrix} a & 2c \\ -n & m \end{vmatrix} = 1$ and $\begin{pmatrix} a & 2c \\ -n & m \end{pmatrix} \begin{pmatrix} m \\ n \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$. If m is even, then we just use the fact that $(m, n) = 1$ to get integers b and d with $bm + dn = 1$. Then $\begin{vmatrix} n & -m \\ b & d \end{vmatrix} = 1$ and $\begin{pmatrix} n & -m \\ b & d \end{pmatrix} \begin{pmatrix} m \\ n \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$. Both matrices are realizable by extensions of homeomorphisms of S . Thus up to homeomorphisms of B , there are only two simple closed curves on ∂B to which to attach a meridian of a fibered solid torus.

From the last paragraph, there are at most two topological spaces that result from sewing a solid torus to B . There are many more than two Seifert fibered spaces that result from sewing a fibered solid torus to the trivial circle bundle over the Möbius band.

We see what we get when we sew a solid torus with meridian sewn to the curves from either of the two classes that we have found. Note that sewing a meridian along along the curve $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ does not give a Seifert fibered space since the curve $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ is homologous to a fiber. We can use the curve $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ instead which is in the same class when we wish to look at the space as a Seifert fibered space. The other class is represented by $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$.

Sewing a meridian of a solid torus to $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ sews a disk to each of the boundary components of the annulus in B containing J . This creates a non-separating S^2 in M . Cutting along this S^2 cuts B into $S^1 \times I \times I$ from which B is recovered by sewing $S^1 \times I \times \{0\}$ to $S^1 \times I \times \{1\}$ with a homeomorphism that takes each $(x, t, 0)$ to $(x, 1 - t, 1)$. The cut along S^2 cuts the solid torus into two cylinders that are sewn to $S^1 \times I \times I$ along $S^1 \times \{0, 1\} \times I$. It is seen that M is the non-orientable S^2 bundle over S^1 .

To view M as a Seifert fibered space, we can sew the meridian of the solid torus along $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ instead. This is a crossing curve for the fibers of M in ∂B so the resulting space has no exceptional fibers. The curve has intersection number one with the boundary K' of the section of the projection from B to its orbit surface A viewing B as a Seifert fibered space. Thus the obstruction to a section for M is 1. Since any curve $\begin{pmatrix} m \\ n \end{pmatrix}$ can be used with m odd to get the same topological space and n is the index of the fiber that results, we can realize M with one exceptional fiber of any given index. Note that the presentation for $\pi_1(M)$ gotten by using the curve $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ is $\langle h, x \mid xhx^{-1}h^{-1} = x^2h^1 = 1 \rangle$ or $\langle h, x \mid x^2 = h^{-1} \rangle$ which is just \mathbf{Z} .

The other space that can be obtained is gotten by sewing a meridian to $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$. This is a crossing curve for the fibers of M in ∂B , so this realizes M as a space with no exceptional fibers. It attaches a disk to the boundary of the section for B so that it gives a section to all of M . Since there are no exceptional

fibers and B is $A \times S^1$, this realizes M as $P^2 \times S^1$. A check of $\pi_1(M)$ gives $\langle h, x \mid xhx^{-1}h^{-1} = x^2h^0 = 1 \rangle$ or $\mathbf{Z} \oplus \mathbf{Z}_2$ which is consistent with our description of M . Since $\pi_1(M)$ is not a non-trivial free product, M is irreducible. It is not P^2 -irreducible since it has a two sided copy of P^2 . We can also realize $P^2 \times S^1$ as a Seifert fibered space with a single exceptional fiber of any index. We have shown:

LEMMA 2.3.8. *The only non-orientable topological spaces that can be obtained as Seifert fibered spaces over P^2 with less than two exceptional fibers are $P^2 \times S^1$ and the twisted S^2 bundle over S^1 . Neither space is aspherical as the first space is irreducible but not P^2 irreducible, and the second space is not irreducible. Both spaces can be realized with no exceptional fibers or with one exceptional fiber of any integer index. ■*

Note that both of these spaces have infinite fundamental group. One has fundamental group $\mathbf{Z} \oplus \mathbf{Z}_2$ and the other has fundamental group \mathbf{Z} . To see what the fate of an ordinary fiber is in these spaces, we have to calculate the fundamental groups using the curves J, K, J', K' described above. We can take J to be the fiber of B seen as an S^1 -bundle over the Möbius band A . This is the fiber structure inherited from M . The meridian of the fibered solid torus will be attached along the simple closed curve $C = mJ' + nK'$ on S' with $(m, n) = 1$. Since J and K generate $H_1(B) = \mathbf{Z} \oplus \mathbf{Z}$, we want to express the curve in terms of J and K and get $C = mJ + 2nK$. If m is odd, then $(m, 2n) = 1$ and the resulting fundamental group is \mathbf{Z} . If m is even, then $(m, 2n) = 2$ and the resulting fundamental group is $\mathbf{Z} \oplus \mathbf{Z}_2$. The only way for the fiber J to be torsion in the result is for n to be zero. This is not an allowable sewing however, since this would put a meridian of the attached fibered solid torus homologous to a fiber. In all of the above, modding out by the subgroup generated by the fiber yields the cyclic group \mathbf{Z}_{2n} . We can summarize.

LEMMA 2.3.9. *A non-orientable Seifert fibered space with orbit surface P^2 and with less than two exceptional fibers has fundamental group \mathbf{Z} or $\mathbf{Z} \oplus \mathbf{Z}_2$. In all cases the subgroup generated by an ordinary fiber is infinite cyclic and the corresponding Fuchsian quotient is finite cyclic of non-zero even order. ■*

We now assume that S is a Klein bottle. This means that the classifying homomorphism of M is the only non-trivial homomorphism available with P^2 . Thus M and B are orientable and B is the orientable I -bundle over the Klein bottle. This fibers as a Seifert fibered space in two ways. One is the fibration inherited from M which has the Möbius band A as orbit space. The other uses curves parallel to the orientation reversing curves of S that has a disk as orbit space with two exceptional fibers of index two. In spite of the fact that this is not the fibering that we are presented with we will use it. One reason is that we are trying to extract topological information and it does not matter which fibering we use. The other is that it leads to structures that we have already analyzed.

We must be careful since adding a fibered solid torus to B does not allow a meridian of the of the fibered solid torus to be homologous to a fiber of B . Thus the fiber of the new fibering must be treated separately since it is not a valid sewing curve in the new fibering but is allowed in the original. Also, the fiber of the original fibering must not be used even though it is allowed in the new fibering.

Adding a fibered solid torus to the new fibering gives orbit surface S^2 with two or three exceptional fibers. These spaces are already known to be irreducible. They have Fuchsian quotient a triangle group $\Gamma(2, 2, r)$ if there is a third exceptional fiber, and are lens spaces if there are only two exceptional fibers.

If we add a fibered solid torus with meridian sewn to the fiber of the new fibering, then the meridian is a crossing curve of the original fibering. In fact the meridinal disk completes a section of the orbit surface of M . Thus M is the orientable S^1 bundle over P^2 which we have previously identified as $P^3 \# P^3$. This has 0 obstruction to a section and its fundamental group has the standard presentation $\langle x, h \mid xhx^{-1} = h^{-1}, x^2 = 1 \rangle$ of D_∞ or $\mathbf{Z}_2 * \mathbf{Z}_2$. The cyclic subgroup generated by an ordinary fiber is infinite and the corresponding Fuchsian quotient is $\mathbf{Z}_2 \oplus \mathbf{Z}_2$.

Note that only one sewing yields $P^3 \# P^3$. When we note that $P^3 \# P^3$ does not come up in any of the other sewing of Seifert fibered spaces that are not P^2 -irreducible, then we will know that the fiber structure of $P^3 \# P^3$ is unique.

The forbidden sewing of the original fibering would add a solid torus along a crossing curve in the new fibering. Thus there would be only two exceptional fibers and the space would be a lens space. The curve would be one boundary component of an annulus which is an I -bundle over an orientation preserving curve of the Klein bottle S . Thus a meridinal disk of the added solid torus and a parallel copy of this disk in the solid torus would create a sphere with the annulus in B . Since the annulus does not separate B , the sphere would not separate the lens space. Thus the forbidden lens space is $S^2 \times S^1$.

We can show that $S^2 \times S^1$ does not arise from any other sewing. The sewing must have no third exceptional fiber, so a crossing curve must be used as the image of a meridian. The sewing is then determined by the invariant b . The fundamental group of a sewing with no third exceptional fiber is presented as

$$\begin{aligned} \langle h, c_1, c_2 \mid c_1 h c_1^{-1} &= c_2 h c_2^{-1} = h, \\ c_1^{\mu_1} h^{\beta_1} &= c_2^{\mu_2} h^{\beta_2} = 1, \\ c_1 c_2 h^b &= 1 \rangle. \end{aligned}$$

We have $c_2 = c_1^{-1}h^{-b}$ and the fact that c_2 commutes with h follows from this and the fact that c_1 commutes with h . We also know that the indexes of the two exceptional fibers are both two, so $\mu_1 = \mu_2 = 2$ and $\beta_1 = \beta_2 = 1$. Thus we are left with an abelian group generated by c_1 and h with relations $2c_1 + h = 0$ and $2c_1 + (2b - 1)h = 0$. The only way for the group to be \mathbf{Z} is for the relations to be dependent. This only happens when $b = 1$. Thus there is only one sewing that yields $S^2 \times S^1$ and it is not one that agrees with a fibration over P^2 with no more than one exceptional fiber.

We can determine which lens spaces are obtainable. First we see which are possible, and then we show that they are realized. Two solid tori are sewn together by a homeomorphism of their boundaries. We assume that the solid tori are oriented and have generators for their first homologies so that the fibers are given as vectors $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$. The action of the homeomorphism is given by $\begin{pmatrix} q & r \\ p & s \end{pmatrix}$. Since we want an orientable result, the homeomorphism should reverse the orientations of the boundaries. Thus $\begin{vmatrix} q & r \\ p & s \end{vmatrix} = -1$. (This is not essential since fibered solid tori determined by $1/2$ admit a fiber preserving, orientation reversing self homeomorphism.) We have

$$\begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} q & r \\ p & s \end{pmatrix} \begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} q + 2r \\ p + 2s \end{pmatrix}.$$

Multiplying both sides of $q + 2r = 1$ by s and both sides of $p + 2s = 2$ by r and subtracting gives $qs - pr = s - 2r$ which equals -1 because of the restriction on the determinant. Solving for s leads to $p = 4 - 4r$ and $q = 1 - 2r$. Letting $n = 1 - r$ gives $p = 4n$ and $q = 2n - 1$. Thus the possible lens spaces are the spaces $L_{4n, 2n-1}$. We will show that they are all realized by showing that all the possible fundamental groups are realized. This will suffice since each fundamental group is represented by exactly one space among the possibilities. We know that the fundamental groups realized are abelian, generated by c_1 and h and have relations $2c_1 + h = 0$ and $2c_1 + (2b - 1)h = 0$. So $h = -2c_1$ and we

get $4c_1(1-b) = 0$. This realizes all cyclic groups of order a multiple of four and all the $L_{4n,2n-1}$ can be achieved.

LEMMA 2.3.10. *If M is an orientable, Seifert fibered space with orbit surface P^2 and less than two exceptional fibers, then M is homeomorphic either to a lens space of type $L_{4n,2n-1}$, or to a Seifert fibered space with orbit surface S^2 and three exceptional fibers with two of index 2, or to the connected sum of two copies of P^3 . All of the fundamental groups are finite except for the fundamental group of $P^3\#P^3$. An ordinary fiber has infinite order in $P^3\#P^3$ and the corresponding Fuchsian quotient is $\mathbf{Z}_2 * \mathbf{Z}_2$.*

2.3.3. Summary of “small” Seifert fibered spaces. We identify certain Seifert fibered spaces as “small.” The list that follows will mention these spaces in overlapping groups. The “small” Seifert fibered spaces are: fibered solid tori; lens spaces which include proper lens spaces (all except S^3 and $S^2 \times S^1$), the irreducible lens spaces (all except $S^2 \times S^1$), and the non-simply connected lens spaces (all except S^3); the two S^2 -bundles over S^1 ; $P^2 \times S^1$; $P^3\#P^3$; and the platonic Seifert fibered spaces (Seifert fibered spaces with orbit surface S^2 and exactly three exceptional fibers of indexes $(2, 2, r)$, $(2, 3, 3)$, $(2, 3, 4)$ or $(2, 3, 5)$).

We have proven the following.

THEOREM 2.3.11. *A compact, connected Seifert fibered space M that is not “small” is P^2 -irreducible, boundary irreducible, aspherical, has torsion free π_1 , has an incompressible surface with infinite fundamental group, and $\pi_1(M)$, $\langle h \rangle$ and $\pi_1(M)/\langle h \rangle$ are all infinite where h is represented by some ordinary fiber. ■*

The exceptions to the various properties are listed below.

Not boundary irreducible: fibered solid tori.

Not P^2 -irreducible: Both S^2 -bundles over S^1 ; $P^2 \times S^1$; $P^3\#P^3$.

Not aspherical: All “small” Seifert fibered spaces except the fibered solid tori.

Torsion in π_1 : Platonic Seifert fibered spaces; irreducible lens spaces; $P^2 \times S^1$; $P^3 \# P^3$.

Finite π_1 : Platonic Seifert fibered spaces; irreducible lens spaces.

Finite $\langle h \rangle$: Same as finite π_1 .

Finite Fuchsian quotient: All “small” Seifert fibered spaces.

No incompressible surfaces other than an S^2 bounding no 3-cell or a properly embedded non-boundary parallel disk: All “small” Seifert fibered spaces.

Thus an adequate definition of a “small” Seifert fibered space could be that the subgroup of π_1 generated by an ordinary fiber is of finite index in π_1 .

Note that this consideration of cases has led to a result.

LEMMA 2.3.12. *Let M be a compact, connected Seifert fibered space and let $\langle h \rangle$ be a cyclic subgroup of $\pi_1(M)$ generated by an ordinary fiber. Then $\langle h \rangle$ is infinite if and only if $\pi_1(M)$ is infinite.* ■

2.4. Relating the fundamental group to the topology

In this section we try to discover how much of the fiber structure of a Seifert fibered space M is determined by the topology of the space. The main problem is to try to identify a fiber from the topology of M . The first step in this is to identify the element of $\pi_1(M)$ represented by a fiber. To do this we have to identify the behavior of an element of $\pi_1(M)$ that is represented by a fiber that is different from behavior of other elements of $\pi_1(M)$.

We know that an element h represented by an ordinary fiber generates a cyclic normal subgroup. From this it follows that the centralizer of h has index no more than two in $\pi_1(M)$. Thus h commutes with at least half the elements in $\pi_1(M)$. However, $\pi_1(M)$ is far from abelian for most Seifert fibered spaces. The quotient $\pi_1(M)/\langle h \rangle$ is a Fuchsian group which has as a quotient the fundamental group

of a surface. For most surfaces, centralizers of elements of their fundamental groups are small. This suggests that the cyclic normal subgroup $\langle h \rangle$ might be distinguished in $\pi_1(M)$ by its algebraic properties.

2.4.1. Fuchsian groups. Let H be a Fuchsian group. It has presentation given by either 2.5 or 2.6. A complex X that has $\pi_1(X) \cong H$ can be obtained by modifying a surface.

If the presentation of H is given by 2.5, then let Y be an orientable surface of genus g and $n + m$ boundary components. Let c_1, \dots, c_n denote the first n boundary components, let E_1, \dots, E_n denote pairwise disjoint disks that are disjoint from Y , let $h_i : \partial E_i \rightarrow c_i$ be a map of degree μ_i , and let X be formed by attaching the E_i to Y using the maps h_i . The standard calculation of a presentation for $\pi_1(X)$ gives the presentation 2.5. The unused boundary components of Y correspond to the elements d_i in the presentation.

If the presentation of X is given by 2.6, then the same construction is done by starting with a non-orientable surface.

The surface Y in the above construction is compact and connected. We can enlarge the notion of a Fuchsian group to include the fundamental group of any complex constructed as above, requiring only that Y be a connected surface.

It pays to loosen the requirements even more by allowing any number of disks to be attached to a given component of the boundary of the surface. We argue that this does not enlarge the class of groups considered. Let J be a boundary component of the surface to which several disks are attached and let α be the element of the fundamental group that J represents. Each disk attached along J guarantees that some power d_i of α is the identity. Let d be the greatest common divisor of the d_i . (This makes sense even if the number of attached disks is infinite.) Then $\alpha^d = 1$ is a consequence of the relations $\alpha^{d_i} = 1$, and each $\alpha^{d_i} = 1$ is a consequence of the relation $\alpha^d = 1$. Thus removing all the

disks attached to J and replacing them with one disk attached by a map of degree d , yields a space with the same fundamental group. If this is done for every boundary component of Y , then we obtain a Fuchsian complex with the same fundamental group as the original group and with no more than one disk attached to each boundary component of Y .

We thus define a "*Fuchsian complex*" to be the space obtained from a connected surface by attaching disks to the surface using maps of non-zero degree from the boundaries of the disks to the boundary components of the surface. (Of course the requirement that the maps be of non-zero degree forces the disks to be attached to compact boundary components of the surface.) A "*Fuchsian group*" is the fundamental group of a Fuchsian complex. This definition makes the following immediate.

LEMMA 2.4.1. *A cover of a Fuchsian complex is a Fuchsian complex, and a subgroup of a Fuchsian group is a Fuchsian group.* ■

We have shown that a Fuchsian group can be represented by a Fuchsian complex X based on a surface Y in which each component of ∂Y has no more than one disk attached to it. Further, if there is a disk attached by a map of degree one, then it can be attached by a homeomorphism. In this case, the disk can be regarded as part of Y and the attaching curve removed from the set of boundary components of Y . Thus every Fuchsian group can be represented by a Fuchsian complex in which no more than one disk is attached to any boundary component of the base surface and in which every attaching map has degree at least two. Such a complex is said to be "*simplified*".

If X is a Fuchsian complex based on a connected surface Y , then we define the boundary of X to be all of the boundary of Y to which no disks are attached. Note that this is not an invariant of the group since we can remove all of the boundary of X and have the same fundamental group.

LEMMA 2.4.2. *Let X be a finite Fuchsian complex with non-empty boundary based on a surface Y . Then $\pi_1(X)$ is a free product with one factor a (possibly trivial) free group and all other factors finite cyclic. If X is simplified, if J is a component of $\partial Y - \partial X$, and if d is the degree of the attaching homeomorphism for the disk in $X - Y$ attached to J , then the element of $\pi_1(X)$ represented by J has order exactly d .*

PROOF The complex can be assumed to be simplified for all parts of the statement. The surface Y has boundary components J_1, \dots, J_n to which disks are attached, and boundary components K_1, \dots, K_m to which no disks are attached. By hypothesis, $m \neq 0$. There are pairwise disjoint, properly embedded arcs $\alpha_1, \dots, \alpha_n$ in Y with each $\partial\alpha_i$ having one point on K_1 and one point on J_i . A regular neighborhood A_i of $J_i \cup \alpha_i$ in Y is an annulus with J_i as one boundary component, and with frontier in Y a properly embedded arc. The closure Y' of $Y - \bigcup(A_i)$ is a surface with boundary. Then $X - Y'$ is a disjoint union of Fuchsian complexes B_1, \dots, B_n where each B_i is made from the annulus A_i by attaching to J_i the disk attached to J_i in X . The results follow immediately. ■

If X is a Fuchsian complex based on a surface Y and S is a set of compact boundary components of Y to which disks are attached, then we can form an identification space of X by taking to a point each component in the collection S and all the disks that are attached along that component. We refer to this identification as "modding out the disks on S ". If S is the collection of all components of ∂Y that have disks attached then we see that modding out the disks on S gives a surface whose boundary is the boundary of X .

LEMMA 2.4.3. *Let X be a Fuchsian complex based on a surface Y and let S be a set of components of ∂Y to which disks have been attached. Then the identification that mods out the disks on S has image a Fuchsian complex and*

induces a surjection on π_1 whose kernel is normally generated by the loops in S . If $\pi_1(X)$ is torsion free, then the identification induces an isomorphism.

PROOF The image of each collection of disks attached to a curve in S is a point. Thus the image of Y in the identification is a surface Y' in which each curve in S has become a point, and the image X' of X is a Fuchsian complex based on the surface Y' . Let x_1, \dots, x_n be the image points of the curves in S . A loop in X' can be homotoped to miss the x_i and we see that the identification induces a surjection on π_1 . A loop in the kernel can be homotoped in X to miss all attached disks and thus reside in Y . Its image in Y' bounds a singular disk that may contain points x_i in its interior. Thus the original loop in Y bounds a singular disk with holes whose other boundary components are elements of S .

If $\pi_1(X)$ is torsion free, then each curve in S must represent the trivial element in $\pi_1(X)$. The kernel of the homomorphism on π_1 must then be trivial. ■

COROLLARY 2.4.3.1. *A torsion free Fuchsian group is a surface group.*

PROOF Let X be a Fuchsian complex based on a surface Y , with $\pi_1(X)$ torsion free. If we mod out the all the attached disks, then we do not change the fundamental group. Modding out all the attached disks gives a surface. ■

COROLLARY 2.4.3.2. *If a finite Fuchsian complex X without boundary has $\pi_1(X)$ torsion free, then $\pi_1(X)$ is the fundamental group of a closed surface.*

PROOF Modding out the attached disks gives a compact surface without boundary whose fundamental group is isomorphic to $\pi_1(X)$. ■

COROLLARY 2.4.3.3. *If a finite Fuchsian complex X with boundary has $\pi_1(X)$ torsion free, then $\pi_1(X)$ is free.*

PROOF Modding out the attached disks gives a compact surface with boundary whose fundamental group is isomorphic to $\pi_1(X)$. ■

We say that a Fuchsian group is "orientable" if it is the fundamental group of a Fuchsian complex that is based on an orientable surface Y . There are non-orientable Fuchsian groups.

LEMMA 2.4.4. *A Fuchsian group with an infinite subgroup isomorphic to the fundamental group of a closed, non-orientable surface is non-orientable.*

PROOF Let X be a Fuchsian complex based on a surface Y and let $\pi_1(X)$ have an infinite subgroup G isomorphic to the fundamental group of a closed, non-orientable surface S . Let \tilde{X} be the cover of X corresponding to G . It is a Fuchsian complex based on the surface \tilde{Y} , the pre-image of Y in \tilde{X} . Since G is torsion free, the curves in $\partial\tilde{Y}$ with attached disks must represent the trivial element in G . Modding out the attached disks must result in a surface with fundamental group G . This surface must be non-orientable, so \tilde{Y} and Y must be non-orientable. ■

LEMMA 2.4.5. *Let M be a Seifert fibered space, let h be represented by an ordinary fiber, and let $p : M \rightarrow G$ be the projection to the orbit surface. Then there is a Fuchsian complex X with an isomorphism $i : \pi_1(M)/\langle h \rangle \rightarrow \pi_1(X)$ so that if $q : X \rightarrow X'$ is the identification map obtained by modding out the attached disks, then there is a homeomorphism $f : X' \rightarrow G$ taking the images of the attached disks to the exceptional points of G so that the following commutes.*

$$\begin{array}{ccccc}
 \pi_1(M) & \longrightarrow & \pi_1(M)/\langle h \rangle & \xrightarrow{i} & \pi_1(X) \\
 p\# \downarrow & & & \swarrow q\# & \\
 \pi_1(G) & \xleftarrow{f\#} & \pi_1(X') & &
 \end{array}$$

PROOF The presentation 2.3 or 2.4 for $\pi_1(M)$ is built from the orbit surface G from which disk neighborhoods of the exceptional points and a single disk neighborhood of an ordinary point have been removed. Let the resulting surface be G' , let c_1, \dots, c_n be the boundary components of G' that are also the boundaries of the disk neighborhoods of the exceptional points, and let e be the boundary component of G' that is also the boundary of the removed disk neighborhood of the ordinary point. The presentation of the Fuchsian quotient 2.5 or 2.6 is obtained if disks are attached to the curves c_i by maps of degree μ_i on the boundaries and a disk is attached to e by a homeomorphism of the boundary. The commutativity of the diagram can now be checked. ■

COROLLARY 2.4.5.1. *If M is a Seifert fibered space, h is represented in $\pi_1(M)$ by an ordinary fiber, and $\pi_1(M)/\langle h \rangle$ is a non-orientable Fuchsian group, then the orbit surface of M is non-orientable.*

PROOF This follows from the previous lemma. ■

The assumption in the next lemma that $\pi_1(X)$ be infinite is seen necessary by considering a Fuchsian complex based on a Möbius band to which a single disk has been added to the boundary with a map of degree d . The centerline of the Möbius band represents an element of order $2d$ and is not conjugate to a power of the boundary of the Möbius band.

LEMMA 2.4.6. *Let X be a simplified Fuchsian complex constructed from a surface Y and assume that $\pi_1(X)$ is infinite. If $\alpha \neq 1$ is an element of $\pi_1(X)$ of finite order, then a conjugate of α is represented by a loop in $\partial Y - \partial X$. Two different components of $\partial Y - \partial X$ represent elements that are not conjugate in $\pi_1(X)$. Also, an element of $\pi_1(X)$ represented by a boundary component of X has infinite order.*

PROOF In proving the first two assertions we can assume that the boundary of X is empty. Let C be the cyclic subgroup generated by α . The cover \tilde{X} corresponding to C is determined by the conjugacy class of C . Thus we must show that some loop in the boundary of Y lifts to \tilde{X} .

Since C is finite and $\pi_1(X)$ is infinite, \tilde{X} is an infinite sheeted cover. Thus \tilde{Y} , the preimage of Y is non-compact. Modding out all attached disks gives a connected, non-compact surface with empty boundary and whose fundamental group is the image of a finite cyclic group. Thus this surface is an open disk and \tilde{Y} is planar. We know that each component J of $\partial\tilde{Y}$ is torsion in the subspace of \tilde{X} formed by J and the disks attached to it. We thus see that $\pi_1(\tilde{X})$ is the free product of cyclic groups. But $\pi_1(\tilde{X})$ is cyclic, so exactly one of the free factors is non-trivial and exactly one component of $\partial\tilde{Y}$ carries the fundamental group of \tilde{X} . This finishes the first assertion.

To consider the second assertion, let α and β be elements represented by two components J_1 and J_2 respectively of $\partial Y - \partial X$. We again consider the cover \tilde{X} corresponding to the cyclic group C generated by α . If β is conjugate to α , then both J_1 and J_2 have lifts to \tilde{X} . But this would give $\pi_1(\tilde{X})$ the structure of a non-trivial free product of cyclic groups.

To consider the third assertion, we return to the original Fuchsian complex X without discarding any of its boundary components. Let J be a component of ∂X . If we mod out all the attached disks, we get a surface with boundary and J is one of the boundary components. If the quotient is a disk, then $\pi_1(X)$ is a free product of more than one cyclic group (since $\pi_1(X)$ is infinite), and J is the product of the generators. But this is an element of infinite order. If the quotient surface is not a disk, then its fundamental group is infinite and J represents an element of infinite order in it. Thus J represents an element of infinite order in $\pi_1(X)$. ■

Note that the “small” Seifert fibered spaces are those with finite Fuchsian quotient. This accounts for the hypothesis in the following consequence of the above lemma.

LEMMA 2.4.7. *Let S be a Seifert fibered space that is not one of the “small” spaces. Let α be an element of $\pi_1(S)$ so that a power of α is homotopic to a power of an ordinary fiber. Then a conjugate of α is represented by a power of an exceptional fiber. If we assume in addition that α is represented by a loop J in a torus boundary component C of S , then J is homotopic in C to a power of a fiber in C .*

PROOF As usual let H_i be the exceptional fibers and N_i be pairwise disjoint fibered solid torus neighborhoods of the H_i in S . Let S' be obtained from S by removing the interiors of the N_i . Let h be represented by an ordinary fiber. The Fuchsian complex for $\pi_1(S)/\langle h \rangle$ is obtained from the orbit surface G' of S' by attaching disks to the curves J_i that are the images of the ∂N_i . The hypothesis says that α maps to a torsion element in $\pi_1(S)/\langle h \rangle$. Since the Fuchsian quotient is infinite, Lemma 2.4.6 says that the image of α is conjugate to a loop in one of the J_i . Thus α is conjugate mod $\langle h \rangle$ to a loop in one of the ∂N_i . But h can be represented by a loop in that ∂N_i so α is conjugate to a loop in that ∂N_i . A loop in ∂N_i is either trivial or a power of the centerline of N_i .

If α is represented by a loop in a boundary component C of S , then Lemma 2.4.6 says that the image of α in $\pi_1(S)/\langle h \rangle$ is either trivial in the image of $\pi_1(C)$, or has infinite order. The hypothesis rules out the latter. The homomorphism from $\Pi_1(C)$ is the homomorphism from $\mathbf{Z} \oplus \mathbf{Z}$ to \mathbf{Z} that kills the factor generated by an ordinary fiber. This completes the proof. ■

The next result is proven by R. H. Fox in “On Fenchel’s conjecture about F-groups,” *Matematisk Tidsskrift*, B (1952), 61–65. It will be used to prove the lemmas that follow about torsion in fuchsian groups.

THEOREM 2.4.8. *Let $p \geq 2$, $q \geq 2$ and $r \geq 2$ be three integers. Then there is a finite group with elements a and b so that the orders of a , b and ab are exactly p , q and r respectively. ■*

LEMMA 2.4.9. *Let X be a simplified Fuchsian complex based on a surface Y with $\pi_1(X)$ infinite. Let J be a component of $\partial Y - \partial X$ and let d be the degree of the attaching homeomorphism for the disk in $X - Y$ attached to J . Then the element of $\pi_1(X)$ represented by J has order exactly d .*

PROOF If X has boundary, then the result is proven above. We assume that X has no boundary.

If there is a non-separating, two sided simple closed curve K in Y , then splitting X and Y along K gives a Fuchsian complex X' based on a surface with at least two boundary components and infinite fundamental group so that $\pi_1(X')$ is a free product of an infinite free group and a collection of finite groups that includes \mathbf{Z}_d generated by J . The two copies of K in X' would represent elements of infinite order in $\pi_1(X')$. Now $\pi_1(X)$ is an HNN extension into which $\pi_1(X')$ includes.

If there is a separating simple closed curve K in Y so that the two complementary domains of K in X had infinite π_1 , then a similar argument proceeds with a free product with amalgamation.

If there is no such curve, then modding out the attached disks must result in a 2-sphere or projective plane. Thus Y is a disk with holes or a Möbius band with holes. If the number of holes in the disk is zero or one, or the number of holes in the Möbius band is zero, then $\pi_1(X)$ is finite. If the number of holes in the disk is three, then the result follows from Fox's theorem and if the number of holes is four or more, then a separating curve K can be found fitting the requirements of the paragraphs above. If the number of holes in the Möbius band is one or

more, then we can also find such a curve. (If an arc is run from the boundary of the Möbius band to the boundary of the hole, then the curve sought is the boundary of a regular neighborhood of the union of the arc, the boundary of the Möbius band and the boundary of the hole.) ■

COROLLARY 2.4.9.1. *If M is a Seifert fibered space, h is represented by an ordinary fiber, and $\pi_1(M)/\langle h \rangle$ is non-trivial and torsion free, then M has no exceptional fibers.*

PROOF We know that $\pi_1(M)/\langle h \rangle$ is infinite. We can construct a simplified Fuchsian complex with fundamental group $\pi_1(M)/\langle h \rangle$ from the presentations 2.5 or 2.6 as appropriate. If there are exceptional fibers, then there will be disks attached with homeomorphisms of degree above 1. From the lemma, we know that there will be torsion in the corresponding Fuchsian group. ■

LEMMA 2.4.10. *Every triangle group has a torsion free, normal subgroup of finite index.*

PROOF A finite group obviously has a torsion free, normal subgroup of finite index, so we restrict our attention to infinite groups. Let $\Gamma(p, q, r)$ be an infinite triangle group presented by $\langle x, y \mid x^p = y^q = (xy)^r = 1 \rangle$ and let G be the group given by Fox's theorem. There is a homomorphism h taking x to a and y to b . The only powers of x , y and xy in the kernel of h are the identity element. Now $\Gamma(p, q, r)$ is a Fuchsian group with Fuchsian complex based on a compact, planar surface with three boundary components corresponding to x , y and xy . By Lemma 2.4.6, we know that any torsion element of $\Gamma(p, q, r)$ is conjugate to a power of x , y or xy . Thus no torsion element of $\Gamma(p, q, r)$ is in the kernel of h . Since G is finite, we have that the kernel of h is a torsion free subgroup of finite index in $\Gamma(p, q, r)$. ■

THEOREM 2.4.11. *Every Fuchsian group corresponding to a finite Fuchsian complex contains the fundamental group of a surface as a normal subgroup of finite index.*

PROOF By Corollary 2.4.3.1, we need only find a normal, torsion free subgroup of $\pi_1(X)$ of finite index. If we find a torsion free subgroup of finite index that is not normal, then the intersection of its finitely many conjugates will be torsion free, normal and of finite index. Thus we are free to replace X by a finite cover of X at any time.

Note that if $\pi_1(X)$ is finite then the trivial subgroup satisfies the conclusion. We assume that $\pi_1(X)$ is infinite.

Let X be a finite Fuchsian complex based on a surface Y . If Y is not orientable, then we can consider the orientable double cover of Y . The boundary components of Y are orientation preserving, and lift to the double cover. Thus we can construct a double cover of X based on the orientable double cover of Y . This gives a subgroup of index two in $\pi_1(X)$ whose Fuchsian complex is based on an orientable surface. Thus we assume that Y is orientable.

We also assume that the complex X is simplified so that no more than one disk is attached along the same curve, and every attaching map is of degree at least two.

If we mod out by the attached disks, then we obtain an orientable surface Y' and a finite number of points x_1, \dots, x_n that are the images of the attached disks. We consider cases based on the genus of Y' and the number n .

If $n = 0$, then $X = Y = Y'$ and $\pi_1(X)$ is an infinite surface group and torsion free.

If $n \geq 3$, then there are a finite number of disks D_1, \dots, D_m in Y' so that no x_i is in a ∂D_j , and so that each D_j contains exactly 3 of the x_i . The D_j are not necessarily disjoint and some of the x_i will be in more than one D_j . For each j ,

let E_j be the preimage of D_j in X , let F_j be the closure of $X - E_j$, and let Z_j be obtained from X by identifying F_j to a point. Each Z_j is a Fuchsian complex based on a disk with two holes and its fundamental group is a triangle group.

Each Z_j contains exactly three boundary components of Y . Let p , q and r be the degrees of the maps for the disks attached to these three boundary components. We know that each degree is at least two and we know that the corresponding power of each boundary component is trivial in $\pi_1(X)$. In $\pi_1(Z_j)$, p , q and r are exactly the orders of the three boundary components. Thus the only power of any of these boundary components that is in the kernel of the induced homomorphism from $\pi_1(X)$ to $\pi_1(Z_j)$ is the identity element. By Lemma 2.4.10, there is a torsion free, normal subgroup of finite index in $\pi_1(Z_j)$. Let N_j be the preimage of this subgroup in $\pi_1(X)$. The subgroup N_j is of finite index in $\pi_1(X)$. Note that no power other than the identity of the three relevant components of ∂Y are in N_j .

Let N be the intersection of the N_j . The subgroup N is of finite index in $\pi_1(X)$. Let α be a torsion element of $\pi_1(X)$. By Lemma 2.4.6, α is conjugate to a power of a component of ∂Y to which a disk is attached. Thus α is not in some N_j and not in N . Thus N is torsion free.

We now assume that $0 < n < 3$ and consider the possibilities for Y' . If Y' is a disk and $n \leq 1$, then $\pi_1(X)$ is finite cyclic. If Y' is a disk and $n = 2$, then $\pi_1(X)$ is a free product of two cyclic groups. We can either appeal to the fact that a free product of cyclic groups has a free subgroup of finite index, or we can use the following argument. If both cyclic groups are of order two, then $\pi_1(X)$ is D_∞ and we know that it has an infinite cyclic subgroup of index two. Assume one order is at least three. We obtain X from a disk with two holes by attaching a disk D to one hole with a map of some degree d and attaching a disk E to the other hole by a map of degree e with $e \geq 3$. Now $X - E$ has an e -fold

cover obtained from an annulus with e holes by attaching one disk to each hole with a map of degree d . We obtain an e -fold cover \tilde{X} of X by attaching e disks to one of the remaining boundary components with degree one maps. Now \tilde{X} has at least e attaching sites of disks with attaching map of degree more than one and we are done by referring to the case where n was at least three.

If Y' is a sphere with $n = 1$ or $n = 2$, then $\pi_1(X)$ is finite.

Since Y' is compact and orientable, then any possibility for Y' other than a disk or 2-sphere has $\pi_1(Y')$ infinite. Thus Y' has finite sheeted covers of arbitrarily high index. Assuming $n \geq 1$, we can imitate our construction of \tilde{X} above to obtain finite sheeted covers of X with any number of attached disks of degree more than 1 and again refer to the case where n was at least three. ■

The sign of the Euler characteristic of the surface in Theorem 2.4.11 is an invariant of the group. If X is a Fuchsian complex and \tilde{X} is a λ -sheeted cover of X with fundamental group H isomorphic to the fundamental group of the surface S , then we can associate the rational number $\chi(S)/\lambda$ to $\pi_1(X)$. If we similarly obtain another subgroup K of finite index γ in $\pi_1(X)$ and a surface S' with fundamental group K , then $H \cap K$ has index γ in H and index λ in K . There is a surface T that is a γ -fold cover of S and a surface T' that is a λ -fold cover of S' whose fundamental groups correspond to $H \cap K$. Since $\pi_1(T) \cong \pi_1(T')$ and both are infinite, we have that T and T' are homotopy equivalent and $\chi(T) = \chi(T')$. Since Euler characteristics multiply with the sheetedness of covers, $\gamma(\chi(S)) = \chi(T) = \chi(T') = \lambda(\chi(S'))$ and $\chi(S)/\lambda = \chi(S')/\gamma$.

We can compute the sign of the Euler characteristic from the presentation of the Fuchsian group. Let the Fuchsian complex X be based on the surface Y . The surface Y has either g handles or k crosscaps. The surface Y also has n boundary components to which disks are attached, and m boundary components that remain boundary components of X . The Euler characteristic of Y is given

by $\chi(Y) = 2 - s - n - m$ where s is either $2g$ or k . Assume that $\pi_1(X)$ has a torsion free subgroup of index λ . Let \tilde{X} be the cover of X determined by this subgroup. Let \tilde{Y} be the preimage of Y in \tilde{X} . We know that \tilde{Y} is a connected λ -fold cover of Y and has $\chi(\tilde{Y}) = \lambda(2 - s - n - m)$. We get \tilde{X} from \tilde{Y} by attaching disks to some of the boundary components of \tilde{Y} . Let J_1, \dots, J_n be the components of ∂Y to which disks are attached in X . Let d_1, \dots, d_n be the degrees of the respective attaching maps. (We assume that the Fuchsian complex X is simplified.) Each component of the preimage of J_i in \tilde{Y} covers J_i by a d_i -fold covering map. Thus there are λ/d_i components of the preimage of J_i . To complete \tilde{Y} to a surface S that carries the fundamental group of \tilde{X} , we must add exactly one disk (by a homeomorphism of the boundary) to each preimage of each J_i . Thus we will add λ/d_i disks for each J_i . We get

$$\begin{aligned} \chi(S) &= \lambda(2 - s - n - m) + \sum_{i=1}^n \frac{\lambda}{d_i} \\ &= \lambda \left[(2 - s - m) - \sum_{i=1}^n \left(1 - \frac{1}{d_i}\right) \right] \end{aligned}$$

where again s is either $2g$ or k depending on whether Y is orientable or not.

We say that a Fuchsian group has "*positive, negative or zero Euler characteristic*" depending on the Euler characteristic of the surface given by Theorem 2.4.11. Since surfaces of positive Euler characteristic have finite fundamental groups, we know that Fuchsian groups of positive Euler characteristic are finite. We can also describe those Fuchsian groups of zero Euler characteristic.

We let g, k, n, m and d_1, \dots, d_n be as in the previous paragraphs. We let Σ denote

$$\sum_{i=1}^n \left(1 - \frac{1}{d_i}\right)$$

and let s denote $2g$ or k as appropriate. We get Euler characteristic zero when $s + m + \Sigma = 2$. Since $g = 0$ and $k = 0$ both correspond to 2-spheres, we have the following possibilities where "holes" are boundary components that have attached disks.

S^2 with 3 or 4 holes	$\Sigma = 2$
disk with 2 holes	$\Sigma = 1$
annulus	$\Sigma = 0$
torus	$\Sigma = 0$
P^2 with 2 holes	$\Sigma = 1$
Möbius band	$\Sigma = 0$
Klein bottle	$\Sigma = 0$

There are only a finite number of ways to get the values of Σ above. For $\Sigma = 2$, the values of the d_i must be one of the following $(2, 2, 2, 2)$, $(2, 3, 6)$, $(2, 4, 4)$ or $(3, 3, 3)$. For $\Sigma = 1$ the only combination possible is $(2, 2)$. This is of interest because of the following.

LEMMA 2.4.12. *If the fundamental group G of a finite Fuchsian complex has a subgroup isomorphic to fundamental group of a torus or a Klein bottle, then the Euler characteristic of G is zero.*

PROOF Let H be the assumed subgroup. A subgroup of finite index in H is also one of the two possibilities for H . This is seen by considering covering spaces of the corresponding surfaces. Thus the subgroup of finite index in G that is the fundamental group of a surface S has a subgroup that is one of the two possibilities for H . The cover of S corresponding to H must be either a torus or Klein bottle. This forces the Euler characteristic of S to be zero. ■

LEMMA 2.4.13. *Every infinite Fuchsian group contains an element of infinite order.*

PROOF Let X be a Fuchsian complex based on a surface Y . If ∂X is not empty, then we are done by Lemma 2.4.6. Let Y' be obtained by modding out all the attached disks. If Y' is not orientable, then we can pass to a 2-sheeted

cover and assume that Y' is orientable. If $\pi_1(Y')$ is infinite, then it has an element of infinite order and so does $\pi_1(X)$. Thus we can assume that Y' is an orientable, surface without boundary and finite fundamental group. Thus Y' is an open disk or 2-sphere.

If Y' is a disk then $\pi_1(X)$ is a free product of cyclic groups and will be infinite if and only if at least two non-trivial cyclic groups are involved. If at least two non-trivial cyclic groups are involved, then there will be an element of infinite order.

If Y' is a 2-sphere, then X is a finite complex and we are done by Theorem 2.4.11. ■

The next lemma discusses Fuchsian groups that have \mathbf{Z} as a subgroup of finite index. Having \mathbf{Z} as a subgroup of finite index is equivalent to having two ends.

LEMMA 2.4.14. *If the fundamental group G of a finite Fuchsian complex has \mathbf{Z} as a subgroup of finite index, then G is \mathbf{Z} or $\mathbf{Z}_2 * \mathbf{Z}_2$.*

PROOF Let X be a finite, simplified Fuchsian complex with fundamental group G based on a surface Y . Let \tilde{X} be the cover corresponding to the \mathbf{Z} subgroup of finite index. If $\partial\tilde{X}$ is empty, then $\mathbf{Z} = \pi_1(\tilde{X})$ is the fundamental group of a closed surface, so $\partial\tilde{X}$ and ∂X must be non-empty.

The hypothesis implies that the number of ends of $\pi_1(X)$ is two. Since ∂X is non empty, $\pi_1(X)$ is a free product of groups, one of which is free (and perhaps trivial) and all others finite cyclic. The number of ends will be infinite if the free group factor has rank more than one, if there are more than two non-trivial factors, or if there are two non-trivial factors and one of the factors has order more than two. Since $\pi_1(X)$ is not finite, the only possibilities left are those of the conclusion. ■

2.4.2. Centralizers in Seifert fibered spaces. We know that the fundamental group of a Seifert fibered space has a cyclic normal subgroup. When this subgroup is infinite, the action on it by conjugation can only carry a generator to itself or its inverse. Thus the cyclic normal subgroup is central in an index 1 or 2 subgroup in the fundamental group. We can say exactly what this subgroup is.

LEMMA 2.4.15. *Let M be a Seifert fibered space with infinite fundamental group, let h be represented by an ordinary fiber, let $p : M \rightarrow G$ be the projection to the orbit surface, and let $\phi : \pi_1(G) \rightarrow \mathbf{Z}_2$ be the classifying homomorphism. Then the centralizer of h in $\pi_1(M)$ is the kernel of $\phi p_\# : \pi_1(M) \rightarrow \mathbf{Z}_2$.*

PROOF Since $\pi_1(M)$ is infinite, h has infinite order in $\pi_1(M)$ and is not equal to its inverse. From the presentations 2.3 and 2.4, we see that a generator a_i , b_i , or c_i is in C , the centralizer of h in $\pi_1(M)$, if and only if it is in the kernel N of $\phi p_\#$. Also, both N and C contain h . We thus have two subgroups of index 1 or 2 that contain exactly the same generators. The two subgroups now yield the same homomorphisms to \mathbf{Z}_2 and are the same. ■

We would like to know what other elements might have centralizers as large. We need one technical lemma.

LEMMA 2.4.16. *If n is an integer greater than 1, then $\mathbf{Z} \oplus \mathbf{Z}_n$ is not a Fuchsian group.*

PROOF Let X be a Fuchsian complex based on a surface Y with $\pi_1(X) \cong \mathbf{Z} \oplus \mathbf{Z}_n$. We may assume that ∂X is empty. If we mod out the attached disks, we get a surface without boundary whose fundamental group is a quotient of $\mathbf{Z} \oplus \mathbf{Z}_n$. This surface must be S^2 , P^2 or an open disk, annulus or Möbius band.

If the quotient surface is P^2 , then there is a subgroup of index 2 with corresponding surface S^2 . We will show that neither \mathbf{Z} nor $\mathbf{Z} \oplus \mathbf{Z}_n$ can occur if

the quotient surface is S^2 . This will eliminate both S^2 and P^2 . If the surface is S^2 , then as argued before, $\pi_1(X)$ is either finite cyclic or a triangle group or a free product with amalgamation of two non-abelian groups (depending on the number of attaching sites of disks in X). However, all the infinite triangle groups are non-abelian.

If the quotient surface is an open disk, then $\pi_1(X)$ is a free product of finite cyclic groups. This cannot yield $\mathbf{Z} \oplus \mathbf{Z}_n$. If the quotient surface is an open annulus or Möbius band, then $\pi_1(X)$ is the free product of \mathbf{Z} and a collection of finite cyclic groups. Again, $\mathbf{Z} \oplus \mathbf{Z}_n$ cannot be obtained. ■

We can now determine the infinite Fuchsian groups that have non-trivial center.

LEMMA 2.4.17. *The only infinite Fuchsian groups that have non-trivial center are \mathbf{Z} , $\mathbf{Z} \oplus \mathbf{Z}$ or the fundamental group of the Klein bottle.*

PROOF Let G be an infinite Fuchsian group with non-trivial center. We start by showing that G is torsion free. We know that G has an element of infinite order. If it has torsion, then it also has an element of finite order. Let a be an element of the center. Let b be another element of G chosen to have finite order if a has infinite order, and chosen to have infinite order if a has finite order. We know a and b commute since a is in the center, so a and b generate a subgroup of G isomorphic to $\mathbf{Z} \oplus \mathbf{Z}_n$ for some $n > 1$. Since a subgroup of a Fuchsian group is a Fuchsian group, this contradicts the previous lemma.

We now have that G is torsion free and must be the fundamental group of a surface S . If S is a non closed surface, then G is trivial or free. The only infinite such group with center is \mathbf{Z} . If S is a closed, orientable surface of genus at least 2 or a closed, non-orientable surface with at least 3 crosscaps, then G is the free product with amalgamation of one non-abelian free group with another

free group of rank at least one amalgamated over \mathbf{Z} . Since the generator of the amalgamating subgroup does not commute with elements of the non-abelian free group, the group G has trivial center. The only remaining surfaces have fundamental groups listed in the statement of the theorem. ■

The groups in the above lemma show up when looking at the Fuchsian quotient of the fundamental group of the Seifert fibered space. The case in which we will apply the above, we will be modding out by a cyclic central subgroup. The next lemma analyzes what can happen.

LEMMA 2.4.18. *Let G be a group and let N be an infinite, cyclic, central subgroup of G . Assume that G/N is (i) \mathbf{Z} , (ii) $\mathbf{Z} \oplus \mathbf{Z}$, or (iii) the fundamental group of a Klein bottle. Then in case (i), G is $\mathbf{Z} \oplus \mathbf{Z}$, in case (ii), G is $\mathbf{Z} \oplus \mathbf{Z} \oplus \mathbf{Z}$ or N is the entire center of G , and in case (iii), G has an index 2 subgroup isomorphic to $\mathbf{Z} \oplus \mathbf{Z} \oplus \mathbf{Z}$ which contains N .*

PROOF The first case is elementary. In the other cases, G is generated by t a generator of N , and two other elements a and b . We have that t commutes with a and b and that $a^{-1}ba = b^\epsilon t^m$ for some $m \in \mathbf{Z}$ and $\epsilon = 1$ in case (ii) and $\epsilon = -1$ in case (iii). If $m = 0$, then G is isomorphic to $\mathbf{Z} \oplus (G/N)$ which satisfies the conclusions of the two last cases. If $m \neq 0$ in case (iii), then $a^{-2}ba^2 = a^{-1}b^{-1}at^m = bt^{-m}t^m = b$ and the index two subgroup in question is generated by a^2 , b and t . We now assume that $m \neq 0$ in case (ii) and show that every central element of G is in N .

Since t commutes with a and b and $ba = abt^m$, we can pass b over a at the expense of introducing a power of t . Thus we can write any element of G as $a^p b^q t^r$. If this element is central, then so is $a^p b^q$ and a^p commutes with b and b^q commutes with a . We have $a^{-1}b^q a = b^q t^{mq}$ so $t^{mq} = b^{-q} a^{-1} b^q a = 1$. Similarly, $t^{mp} = 1$ is derived from $bab^{-1} = at^m$. But t has infinite order and we assume $m \neq 0$, so both p and q are 0. Thus the central element $a^p b^q t^r$ is in N . ■

The initial hypotheses in the next lemma could be stated by requiring that S not be “small.”

LEMMA 2.4.19. *Let S be a compact, connected Seifert fibered space with infinite, torsion free fundamental group and infinite Fuchsian quotient. Let G be the orbit surface of S , let h be represented by an ordinary fiber, let C be the centralizer of h in $\pi_1(S)$, and let α not a power of h generate a cyclic normal subgroup of $\pi_1(S)$. Then $\mathbf{Z} \oplus \mathbf{Z}$ or $\mathbf{Z} \oplus \mathbf{Z} \oplus \mathbf{Z}$ is a subgroup of finite index in $\pi_1(S)$ containing $\langle h \rangle$. The possible indexes are given by the following table.*

	<u>$\mathbf{Z} \oplus \mathbf{Z}$</u>	<u>$\mathbf{Z} \oplus \mathbf{Z} \oplus \mathbf{Z}$</u>
<i>no extra assumptions</i>	<i>1, 2 or 4</i>	<i>1, 2, 4 or 8</i>
<i>$C = \pi_1(S)$</i>	<i>1 or 2</i>	<i>1, 2 or 4</i>
<i>G is orientable</i>	<i>1, 2 or 4</i>	<i>1, 2 or 4</i>
<i>$C = \pi_1(S)$ and G is orientable</i>	<i>1 or 2</i>	<i>1 or 2</i>

*If $\mathbf{Z} \oplus \mathbf{Z} \oplus \mathbf{Z}$ is not a subgroup of $\pi_1(S)$ with one of the indicated indexes, then $\pi_1(S)/\langle h \rangle$ is either \mathbf{Z} or $\mathbf{Z}_2 * \mathbf{Z}_2$.*

PROOF Let N be generated by h . We know that N is infinite, and that the index of C in $\pi_1(S)$ is 1 or 2. Similarly, the centralizer A of α is of index 1 or 2 in $\pi_1(S)$. If α is not in C , then no power of h commutes with α , no power of h is in A , and A is not of finite index in $\pi_1(S)$. Thus α is in C . By hypothesis, α is not in N .

Since α is in C , α commutes with h and all its powers, so N is in A . Also, N is in C . Let $\bar{\alpha}$, \bar{A} and \bar{C} be the images of α , A and C in $\pi_1(S)/N$. Since $\pi_1(S)/N$ is infinite, both \bar{A} and \bar{C} are infinite. In particular, \bar{A} is an infinite Fuchsian group with $\bar{\alpha}$ as a non-trivial central element. Thus \bar{A} is one of \mathbf{Z} , $\mathbf{Z} \oplus \mathbf{Z}$, or the fundamental group of a Klein bottle.

Let $H = A \cap C$ and let its image in $\pi_1(S)/N$ be \bar{H} . We know that N is in H and is central in it. Also, H is of index 1 or 2 in A , and \bar{H} is of index 1 or 2 in \bar{A} . So \bar{H} is also one of \mathbf{Z} , $\mathbf{Z} \oplus \mathbf{Z}$ or the fundamental group of a Klein bottle. We

also know that N is not the entire center of H since $\alpha \notin N$ is central in H . By the previous lemma, H is $\mathbf{Z} \oplus \mathbf{Z}$ or has $\mathbf{Z} \oplus \mathbf{Z} \oplus \mathbf{Z}$ as an index 1 or 2 subgroup containing N . The possibility that $H = \mathbf{Z} \oplus \mathbf{Z}$ only occurs when \overline{H} is \mathbf{Z} . But \overline{H} is a subgroup of finite index in $\pi_1(S)/\langle h \rangle$. This, and lemma 2.4.14 give the last conclusion.

We must calculate indexes. All indexes discussed are powers of 2. The index of C in $\pi_1(S)$ is 1 or 2. The index of H in A is the index of C in $\pi_1(S)$ and the index of A in $\pi_1(S)$ is 1 or 2. Thus the index of H in $\pi_1(S)$ is no more than 4 and if $C = \pi_1(S)$, then it is no more than 2. This accounts for the possible indexes if $H = \mathbf{Z} \oplus \mathbf{Z}$ or $H = \mathbf{Z} \oplus \mathbf{Z} \oplus \mathbf{Z}$. The only way for H not to be $\mathbf{Z} \oplus \mathbf{Z} \oplus \mathbf{Z}$ but have $\mathbf{Z} \oplus \mathbf{Z} \oplus \mathbf{Z}$ as a subgroup of index 2 in H is for \overline{H} to be the fundamental group of a Klein bottle. Then $\pi_1(S)/\langle h \rangle$ is a non-orientable Fuchsian group and the orbit surface of S is non-orientable. This accounts for the remaining possibilities. ■

The last lemma says that for “large” Seifert fibered space, the absence of $\mathbf{Z} \oplus \mathbf{Z}$ and $\mathbf{Z} \oplus \mathbf{Z} \oplus \mathbf{Z}$ as subgroups of the fundamental group implies that the cyclic normal subgroup generated by an ordinary fiber is the unique maximal, cyclic normal subgroup of the fundamental group. The presence of $\mathbf{Z} \oplus \mathbf{Z}$ or $\mathbf{Z} \oplus \mathbf{Z} \oplus \mathbf{Z}$ as subgroups limits the Fuchsian quotient to have Euler characteristic zero. This limits the structure of the Seifert fibered space. The fact that these subgroups are of finite index, says that there is a finite cover of the space having these subgroups as fundamental group. This is even more of a restriction. In the next section we will consider the structure of finite covers of Seifert fibered spaces. Before that we extract a little more information from our knowledge of the fundamental group.

LEMMA 2.4.20. *Let S be a compact, connected Seifert fibered space with infinite, torsion free fundamental group and infinite Fuchsian quotient. Let h be represented by an ordinary fiber, and let α not a power of h generate a cyclic*

normal subgroup of $\pi_1(S)$. Then the orbit surface of S and the exceptional fibers are limited to the following combinations.

<i>orbit surface</i>	<i>exceptional fibers</i>
S^2	four of index 2
D^2	two of index 2
$S^1 \times I$	none
$S^1 \times S^1$	none
P^2	two of index 2
Möbius band	none
Klein bottle	none

PROOF We know that the Fuchsian group $\pi_1(S)/\langle h \rangle$ is built from the orbit surface G of S by removing neighborhoods of the exceptional points and attaching disks with maps of various degrees to the resulting boundaries. The list in the conclusion is the list giving the Fuchsian groups of zero Euler characteristic with some omissions. We must show that $\pi_1(S)/\langle h \rangle$ has zero Euler characteristic and that the omissions are legitimate.

We know that $\pi_1(S)$ has a subgroup of finite index containing $\langle h \rangle$ isomorphic to $\mathbf{Z} \oplus \mathbf{Z}$ or $\mathbf{Z} \oplus \mathbf{Z} \oplus \mathbf{Z}$. In the latter case, $\mathbf{Z} \oplus \mathbf{Z}$ is a subgroup of $\pi_1(S)/\langle h \rangle$ and $\pi_1(S)/\langle h \rangle$ has zero Euler characteristic. In the former case, $\pi_1(S)/\langle h \rangle$ is \mathbf{Z} or $\mathbf{Z}_2 * \mathbf{Z}_2$. These Fuchsian groups can only be built from an annulus or Möbius band with no exceptional points, or from a disk two exceptional points of index two. These are also included on the list of possibilities with zero Euler characteristic.

We must show that the items omitted from the list are not possible under the hypotheses. The omitted items all have orbit surface the 2-sphere, and three exceptional fibers with indexes $(2, 3, 6)$, $(2, 4, 4)$ or $(3, 3, 3)$. Since the orbit surface is S^2 , we are not in the situation analyzed in the previous paragraph where $\mathbf{Z} \oplus \mathbf{Z}$ is a subgroup of finite index in $\pi_1(S)$. The classifying homomorphism is trivial, so h is central in $\pi_1(S)$. Also, S^2 is orientable. Thus $K = \mathbf{Z} \oplus \mathbf{Z} \oplus \mathbf{Z}$ is a subgroup of index 1 or 2 in $\pi_1(S)$. Since h is in K , the image \overline{K} in $\pi_1(S)/\langle h \rangle$

is of index 1 or 2 in $\pi_1(S)/\langle h \rangle$. We cannot have $\overline{K} = \mathbf{Z} \oplus \mathbf{Z} \oplus \mathbf{Z}_n$ since $\mathbf{Z} \oplus \mathbf{Z}_n$ is not a Fuchsian group. Thus $\mathbf{Z} \oplus \mathbf{Z}$ is a subgroup of index 1 or 2 in $\pi_1(S)/\langle h \rangle$. However, $\pi_1(S)/\langle h \rangle$ is the triangle group $\Gamma(2, 3, 6)$, $\Gamma(2, 4, 4)$ or $\Gamma(3, 3, 3)$. These all have torsion elements of order more than 2 and cannot have a torsion free subgroup of index 1 or 2. ■

2.4.3. Covers of Seifert fibered spaces. In the previous section, we saw that certain assumptions about a Seifert fibered space implied that the fundamental group would have certain groups as subgroups of finite index. This has strong implications for the structure of the covers of the Seifert fibered spaces corresponding to these subgroups. This motivates the topic of this section — the structure of covers of Seifert fibered spaces.

Let M be a Seifert fibered space. Finite sheeted covers of M turn out to inherit a Seifert fibered structure from M . The invariants of this fiber structure are computable, and in some cases easily computable.

In this section we will have to be careful and distinguish between lifts and pre-images. If $p : X \rightarrow Y$ is a covering projection and J is a circle in Y , then $p^{-1}(J)$ is the pre-image of J . This may or may not be connected and a given component of $p^{-1}(J)$ may or may not be a circle. We get a lift of J by choosing a point in J as a basepoint and an orientation so that we can regard J as a closed path. A lift of J is a path in X starting at a pre-image of the basepoint that covers J once as a path. It is possible that no lift of J is a circle while some or all components of the pre-image of J are circles. The fact that a component of the pre-image of J is a circle implies that some power of J (regarded as an element of $\pi_1(Y)$) lifts to a closed path in X .

Let $p : \tilde{M} \rightarrow M$ be a covering projection and let H be a fiber in M . Then $p^{-1}(H)$ is a 1-manifold and each component is a line or a circle. This fibers \tilde{M} with lines and circles. Let N be a fibered solid torus neighborhood of H , let H'

be a component of $p^{-1}(H)$ and let N' be the component of $p^{-1}(N)$ that contains H' . The degree of $p|_{N'}$ equals the degree of $p|_{H'}$. This degree is finite precisely when H' is a circle, or equivalently, when N' is a solid torus. Let $d(H')$ denote the degree of $p|_{N'}$.

If the degree of $p|_{N'}$ is finite, then for any fiber of M in N , a component of the pre-image of the fiber that intersects N' lies in N' and is a circle. Thus a component of the pre-image of a fiber that is a circle has a neighborhood of pre-images of fibers that are circles. Similarly, a component of the pre-image of a fiber that is a line has a neighborhood of pre-images of fibers that are lines. Thus in \tilde{M} , the components of pre-images that are circles and the components of pre-images that are lines form disjoint open sets that cover \tilde{M} . If \tilde{M} is connected, one of these sets is empty. We have shown:

LEMMA 2.4.21. *If M is a Seifert fibered space and $p : \tilde{M} \rightarrow M$ is a covering projection with \tilde{M} connected for which one fiber H of M has a component of $p^{-1}(H)$ a circle, then \tilde{M} inherits a Seifert fiber structure from M . ■*

From now on we only consider covers of M where all components of the pre-images of fibers are circles. This happens in finite sheeted covers of Seifert fibered spaces, but can happen in other ways as well. Note that $G \times S^1$ with G a surface with infinite fundamental group is infinitely covered by $R^2 \times S^1$.

Let H now be an ordinary fiber of M and let H' , N and N' be as above. The pre-images of fibers in N' fiber N' as an ordinary solid torus and the restriction of p to each fiber in N' has the same degree as $p|_{H'}$. Thus the function d is constant over all the fibers in N' . We also have that the pre-images in \tilde{M} of the exceptional fibers are isolated. Thus the pre-images in \tilde{M} of the ordinary fibers of M form a connected subset of \tilde{M} which can be covered by pre-images of ordinary solid tori in M . This implies that the function d is constant over all

the pre-images in \tilde{M} of ordinary fibers in M . Let this common value be called the "degree" of p and be denoted $d(p)$.

Now let H be an exceptional fiber of M and let H' , N and N' be as above. Let the fiber structure of N be determined by ν/μ . Let $\sigma = d(H')$. The fiber structure of N is derived from a rotation of the unit disk by $2\pi(\nu/\mu)$. The fiber structure of N' is derived from a rotation of the unit disk by $2\pi\sigma(\nu/\mu)$. Assuming that ν/μ is in reduced terms, the rational number in reduced terms that determines N' is

$$\frac{\sigma\nu/(\sigma, \mu)}{\mu/(\sigma, \mu)}$$

where (σ, μ) is the greatest common divisor of σ and μ . Let $g = (\sigma, \mu)$, so the invariants of N' are $(\sigma\nu/g)/(\mu/g)$.

Let J be a simple closed curve on ∂N . In terms of a meridian-longitude pair (m, l) we can write $J = \alpha m - \beta l$. We write it this way because with $\psi(m, l) = -\psi(l, m) = 1$, we have $J = \psi(J, l)m - \psi(J, m)l$ and $\psi(J, m) = \beta$.

If we think of N as made from one copy of $D \times I$, then we can think of N' as made from σ copies of $D \times I$. We can declare the "length" of a curve J in ∂N to be $\psi(J, m)$, the algebraic intersection of the curve with m . We can declare the "length" of a curve in $\partial N'$ to be the algebraic intersection of the curve with the full pre-image in N' of m . A full pre-image in N' of m will consist of σ disjoint, parallel meridians of N' . This gives J length β in N . The full pre-image of J in N' has length $\sigma\beta$.

The subgroup of $\pi_1(N)$ corresponding to the covering map from N' to N is $\sigma\mathbf{Z}$. Since J represents β times a generator of $\pi_1(N)$, the smallest power of J that lies in $\sigma\mathbf{Z}$ is $[\sigma, \beta]/\beta$ where $[,]$ is the least common multiple. Since $[\sigma, \beta] = \sigma\beta/(\sigma, \beta)$, we have that the smallest power of J that lies in $\sigma\mathbf{Z}$ is $\sigma/(\sigma, \beta)$. Such a power of J will have length $\sigma\beta/(\sigma, \beta)$ in N and a lift of this power of J will be a simple closed curve of the same length in N' . Thus each

component of the pre-image of J in N' has length $\sigma\beta/(\sigma, \beta)$, the full pre-image of J in N' has length $\sigma\beta$, and J is covered by

$$(2.12) \quad (\sigma, \beta)$$

components.

Let J' be a component of the pre-image of J in N' . The intersection of J' with the full pre-image of m is the length of J' and is $\sigma\beta/(\sigma, \beta)$. But m lifts to σ copies of m . Thus the intersection of J' with a meridian m' of N' is $\beta/(\sigma, \beta)$.

The pre-image l' of l in N' is connected, is a longitude of N' and has length σ . The intersection of J with l is α , and a calculation similar to the one just done gives that the intersection of J' with l' is $\alpha\sigma/(\sigma, \beta)$.

Thus the homology class of J' is determined as

$$(2.13) \quad J' = \psi(J', l')m' - \psi(J', m')l' = \frac{\alpha\sigma}{(\sigma, \beta)}m' - \frac{\beta}{(\sigma, \beta)}l'.$$

If we apply 2.12 and 2.13 to an ordinary fiber of form $\nu m + \mu l$ in ∂N , we get that it is covered by (σ, μ) components (which are ordinary fibers in $\partial N'$) and that each component is of the form

$$\frac{\nu\sigma}{(\sigma, \mu)}m' + \frac{\mu}{(\sigma, \mu)}l'.$$

This agrees with our analysis of the type of N' above.

Recall that $g = (\sigma, \mu)$. Ordinary fibers in N are covered by g ordinary fibers in N' . The restriction of p to N' induces a map of the orbit surface of N' to the orbit surface of N . Both surfaces are disks, and off the center point, the map is g to one. Thus the map is a branched cover of degree g with branch point the center of the disk.

Since each ordinary fiber in N is covered by g ordinary fibers in N' , and each ordinary fiber in M is covered $d(p)$ times by each component of its pre-image in M , we have that each ordinary fiber in N is covered $gd(p)$ times by its full pre-image in N' . We get that $gd(p) = \sigma$ and σ is a multiple of $d(p)$.

Let A be the subgroup of $\pi_1(M)$ that corresponds to the covering $p : \tilde{M} \rightarrow M$, and let h be an element of $\pi_1(M)$ determined by an ordinary fiber. Letting $d = d(p)$, we know that h^d is in A and no smaller power is. In other words, $A \cap \langle h \rangle$ is a subgroup of index d in $\langle h \rangle$. If $\langle h \rangle$ is in A , then $d = 1$ and every ordinary fiber lifts to a closed path. Because of the results in the previous section, we will be interested in covers where ordinary fibers lift to closed paths. Let us call such a cover a "primary" cover of M .

If p is a primary covering map, then $d = d(p) = 1$ and for a component of the pre-image H' of an exceptional fiber with σ as above, we have $g = gd = \sigma$ where $g = (\mu, \sigma)$. Thus $\sigma = (\mu, \sigma)$ and $\sigma | \mu$. The induced map of orbit surfaces is a branched cover of degree $g = \sigma$ near the corresponding exceptional point.

We are primarily interested in primary covering maps. This is because the subgroups discussed in the previous section included the cyclic normal subgroup generated by an ordinary fiber. The covers corresponding to such subgroups are primary covers.

Let H be an exceptional fiber with invariant ν/μ , and let N, N', m, σ and $g = (\sigma, \mu)$ be as above. Let Q be a crossing curve on ∂N . Now the meridian $m = \mu Q + \beta H$ is a simple closed curve, so $(\mu, \beta) = 1$. Since the cover is primary, $\sigma | \mu$ and $(\sigma, \beta) = 1$. We have $Q = \alpha m + \beta l$ for some α and from 2.12 and 2.13 we get that the pre-image of Q in N' is connected and has intersection β with a meridian of N' .

We are going to look at two types of covers. One will be a special type of primary cover in which not only the ordinary fibers lift to circles, but all fibers lift to circles. This makes the handling of the crossing curves determined by a fibered solid torus easy. We will see examples of this type of cover when we give two applications. One will be the orientable double cover of a non-orientable Seifert fibered space. The second will be a double cover of a space with a non-orientable orbit surface in which the cover has an orientable orbit surface. The

second type of cover will be one in which there are no exceptional fibers in the covering space. The crossing curves are easy in this case as well. That there are such covers is shown by the following.

LEMMA 2.4.22. *Let M be a Seifert fibered space, let $h \in \pi_1(M)$ be represented by an ordinary fiber and assume that $\pi_1(M)/\langle h \rangle$ is infinite. Then M has a primary finite sheeted cover with no exceptional fibers in the induced fibration.*

PROOF Let $h \in \pi_1(M)$ be represented by an ordinary fiber. The Fuchsian group $\pi_1(M)/\langle h \rangle$ has a torsion free subgroup K of finite index. We write \overline{K} to represent the pre-image in $\pi_1(M)$ of K . Then \overline{K} has finite index in $\pi_1(M)$. The cover \tilde{M} of M corresponding to \overline{K} is a Seifert fibered space and each ordinary fiber of M lifts to an ordinary fiber in \tilde{M} . To avoid extra notation with induced homomorphisms, we regard $\pi_1(\tilde{M})$ as the subgroup \overline{K} of $\pi_1(M)$. If $\tilde{h} \in \pi_1(\tilde{M})$ is represented by an ordinary fiber of \tilde{M} , then $\langle \tilde{h} \rangle = \langle h \rangle$. The Fuchsian quotient $\pi_1(\tilde{M})/\langle \tilde{h} \rangle$ is just $\overline{K}/\langle h \rangle = K$ and is torsion free and infinite. Thus \tilde{M} has no exceptional fibers. ■

Seifert fiberings of the 3-sphere with exceptional fibers show that the hypotheses of the previous lemma are necessary.

COROLLARY 2.4.22.1. *The cover \tilde{M} of the previous lemma can be assumed to be orientable.*

PROOF Fibers are orientation preserving curves in a Seifert fibered space. The orientable double cover of a Seifert fibered space is a primary cover and a primary cover of a primary cover is a primary cover. Lastly, covers cannot introduce exceptional fibers where there were none before. Taking the orientable double cover of the result of the previous lemma gives the result. ■

We now consider a cover of a Seifert fibered space in which all lifts of all fibers are circles. We first assume that the base is oriented. We consider the non-orientable case later when we look only at the orientable double cover.

Let M be a connected, oriented Seifert fibered space and let $p : \tilde{M} \rightarrow M$ be a covering projection as described. We assume that M and \tilde{M} are compact so that the number of sheets λ of the cover is finite. The number σ defined above which gives the degree of the projection on a fiber in the cover is 1 for all fibers in \tilde{M} . We recall more notation for the structure of M . We let H_1, \dots, H_n be the exceptional fibers of M , we let N_1, \dots, N_n be pairwise disjoint fibered solid torus neighborhoods of the H_i with meridians m_1, \dots, m_n , we let $(\mu_1, \beta_1), \dots, (\mu_n, \beta_n)$ be the crossing invariants, and let Q_1, \dots, Q_n be the crossing curves determined by the N_i . We let N_0 with meridian m_0 be an ordinary solid torus neighborhood of some ordinary fiber in M with N_0 disjoint from the N_i . We let Y be the image of a section of the orbit surface of the closure M_0 of $M - (N_0 \cup N_1 \cup \dots \cup N_n)$ with $\partial Y \cap \partial N_i = Q_i$ for $i \geq 1$. We let $Q_0 = \partial Y \cap \partial N_0$. We know that $m_0 = Q_0 + bH$ and $m_i = \mu_i Q_i + \beta_i H$ where H represents an ordinary fiber in the appropriate torus. We let \tilde{M}_0 be the preimage of M_0 in \tilde{M} . It is connected since it is obtained from \tilde{M} by removing the pre-images of the N_i . These are solid tori with connected boundary and if \tilde{M}_0 is not connected, then \tilde{M} cannot be connected. The preimage of Y in \tilde{M}_0 is a surface \tilde{Y} which hits each fiber of \tilde{M}_0 in a single point. Thus \tilde{Y} is connected.

Each N_i , $i \geq 1$, lifts to λ copies of itself that cover N_i once each. Thus each component of $p^{-1}(N_i)$ determines a crossing curve that is a lift of Q_i . This is just a boundary component of \tilde{Y} . Each component of a lift of N_0 has a meridian with intersection b with the appropriate boundary component of \tilde{Y} .

If we remove the lifts of the N_i , $i \geq 1$, from \tilde{M} and replace them with ordinary solid tori with meridians going to the lifts of the Q_i , then we would

be in a position to calculate the obstruction to the section. The partial section would contain the surface \tilde{Y} together with the meridians bounded by the Q_i . If N_0 lifted to a single solid torus, we would be in good position. Unfortunately it lifts to λ different tori. We need a lemma.

LEMMA 2.4.23. *Let M be a closed, connected, oriented Seifert fibered space with no exceptional fibers. Let N_1, \dots, N_λ be pairwise disjoint saturated solid tori in M , let M_0 be the closure of $M - (N_1 \cup \dots \cup N_\lambda)$, and let Y be the image of a section for M_0 with boundary components $Q_i = \partial Y \cap \partial N_i$. Let b_1, \dots, b_λ be the intersections of the meridians m_i of the N_i with the Q_i . Then the obstruction to the section for M is $\sum b_i$.*

PROOF If all the b_i , $i > 1$, are zero, then we are done since Y extends through the N_i , $i > 1$. We will be done by induction when we show that we can replace b_1 by $b_1 + b_2$ and b_2 by zero while leaving the b_i , $i > 2$ the same and obtain a manifold of the same fiber type as M . Let α be an arc in Y from Q_1 to Q_2 . The saturated annulus A over α has two fibers for its boundary. We can orient these fibers to have the “same” direction in A . If we orient ∂N_1 and ∂N_2 consistently with the orientations that N_1 and N_2 inherit from M (the convention from the first chapter), then Q_1 and Q_2 have intersections of opposite sign with the fibers in ∂A . There is a self homeomorphism of M_0 fixed off a neighborhood of A that rotates one “side” of A through b_2 full rotations. The direction can be chosen to take Q_2 to $Q_2 + b_2 H = m_2$. This will take Q_1 to $Q'_1 = Q_1 - b_2 H$. This makes m_2 and Q'_1 the new boundary curves of reference, and the new intersections are calculated from $m_2 = m_2 + 0H$ and $m_1 = Q_1 + b_1 H = Q_1 - b_2 H + b_2 H + b_1 H = Q'_1 + (b_1 + b_2)H$. This completes the proof. ■

We can now prove the following.

LEMMA 2.4.24. *Let M be a compact, connected, oriented Seifert fibered space and let $p : \tilde{M} \rightarrow M$ be a primary, λ -sheeted cover with λ finite. Let \tilde{M} have the induced fiber structure. If all fibers of M lift to circles in \tilde{M} then each exceptional fiber H is covered by λ exceptional fibers of \tilde{M} with the same crossing invariants as H . If M is closed with obstruction to section b , then the obstruction to section for \tilde{M} is λb . The orbit surface of \tilde{M} is a λ -sheeted cover of the orbit surface of M , and the classifying homomorphism for \tilde{M} is obtained by composing the homomorphism induced by the projection with the classifying homomorphism for M .*

PROOF The previous lemma gives the statement about the obstruction to the section. We have demonstrated above the facts about the crossing invariants. The remaining statements are straightforward. \blacksquare

We now consider a non-orientable Seifert fibered space M with n exceptional fibers. We look at the orientable double cover \tilde{M} of M . Since every fiber in M is orientation preserving in M , it lifts only to circles in \tilde{M} . We adopt the notation developed for the case of oriented M . Each exceptional fiber H_i with fibered solid torus neighborhood N_i has crossing invariants (μ_i, β_i) with $0 \leq \beta_i \leq \mu_i/2$. There will be two components of the pre-image of N_i in \tilde{M} so \tilde{M} will have $2n$ exceptional fibers. The covering translation of \tilde{M} is an orientation reversing involution which will carry one component of the pre-image of N_i to the other. Thus one component will have crossing invariants (μ_i, β_i) and the other will have crossing invariants $(\mu_i, \mu_i - \beta_i)$. This determines the exceptional fibers of \tilde{M} and the collection of crossing invariants. It turns out that the obstruction to the section \tilde{b} for \tilde{M} is remarkably easy to calculate. The existence of a fiber preserving, orientation reversing homeomorphism says that \tilde{M} is fiber equivalent to \tilde{M} with the opposite orientation. The result at the end of chapter 1 says that \tilde{b} must be equal to $-2n - \tilde{b}$ since there are $2n$ exceptional fibers in \tilde{M} . Thus $\tilde{b} = -n$ and is independent of the crossing invariants. We have proven.

LEMMA 2.4.25. *Let M be a compact, connected, non-orientable Seifert fibered space with n exceptional fibers and let \tilde{M} be the orientable double cover of M . Then \tilde{M} has $2n$ exceptional fibers. Each crossing invariant pair (μ_i, β_i) for M is replaced by two pairs (μ_i, β_i) and $(\mu_i, \mu_i - \beta_i)$ for \tilde{M} and if M is closed, then the obstruction to a section for \tilde{M} is $-n$. The orbit surface for \tilde{M} is a double cover of the orbit surface for M and the classifying homomorphism for \tilde{M} must be the orientation homomorphism for its orbit surface. ■*

We now consider primary covers with no exceptional fibers. We restrict ourselves to oriented base. We continue our previous notation and keep the same meanings for M , M_0 , \tilde{M} , \tilde{M}_0 , H_i and (μ_i, β_i) for $1 \leq i \leq n$, N_i , Q_i and m_i , for $0 \leq i \leq n$, Y , \tilde{Y} and λ . Each H_i has a set of numbers σ_{ij} where H_{ij} are the components of the pre-image of H_i and σ_{ij} is the degree of the map from H_{ij} to H_i . Since the cover is a λ -sheeted map, we must have $\sum_j \sigma_{ij} = \lambda$ for each i . Each component of the pre-image of N_i is an ordinary solid torus. We let N_{ij} denote the component containing H_{ij} . Since we have a primary cover, the analysis done at the beginning of this section shows that each crossing curve Q_i has one component in its pre-image in each N_{ij} and that it has intersection β_i with a meridian of N_{ij} .

Since we have a primary cover, we know that each $\sigma_{ij} | \mu_i$. However, the index of the fiber H_{ij} is $\mu_i / (\mu_i, \sigma_{ij})$. Since the cover has no exceptional fibers, this must be 1 and (μ_i, σ_{ij}) must be μ_i . Thus $\mu_i | \sigma_{ij}$ and $\sigma_{ij} = \mu_i$ for all j . Thus every N_{ij} is a μ_i -fold cover of N_i and there are λ / μ_i of them. The total intersection of the meridians of the N_{ij} with the lifts of Q_i (boundary components of a section for \tilde{M}_0) is $\lambda \beta_i / \mu_i$. We have previously shown that the total intersection of the meridians of the lifts of N_0 with the boundaries of a section for \tilde{M}_0 is λb . From the lemma above on the obstruction to a section, we have that the obstruction

to a section for \tilde{M} is

$$\lambda b + \sum \frac{\lambda \beta_i}{\mu_i} = \lambda \left(b + \sum \frac{\beta_i}{\mu_i} \right).$$

We can now extend one of the lemmas above.

LEMMA 2.4.26. *Let M be a compact, connected, oriented seifert fibered space with n exceptional fibers with crossing invariants (μ_i, β_i) , $1 \leq i \leq n$. If M is closed, let the obstruction to a section for M be b . Let $h \in \pi_1(M)$ be represented by an ordinary fiber. Let $\pi_1(M)/\langle h \rangle$ be infinite and have a torsion free subgroup of finite index λ . Then M has a λ -sheeted cover \tilde{M} with no exceptional fibers and, if M is closed, it has obstruction to a section given by*

$$\lambda \left(b + \sum \frac{\beta_i}{\mu_i} \right).$$

■

REMARK The quantity

$$b + \sum \frac{\beta_i}{\mu_i}$$

is, in fact, more natural than b itself. This has been exploited in various papers (e.g., Neumann and Raymond, Lecture Notes in Math. 664).

2.4.4. Uniqueness of fibers. We can combine the results of the previous sections to give a statement about the uniqueness of the elements of the fundamental group represented by the ordinary fibers of "any" Seifert fibration of a Seifert fibered space. Such a statement needs restrictions since fiber structures are not unique for certain spaces. Once we understand the uniqueness of the elements of the fundamental group that can be represented by an ordinary fiber, we can give results about the uniqueness of the fiber structure of certain Seifert fibered spaces. We will only do this under the assumption that the spaces have non-empty boundary even though it is possible to prove similar statements about closed manifolds.

To help with the next statement we define certain Seifert fibered spaces as “flat.” They include some closed manifolds and some with boundary. The closed manifolds are those determined by the following:

$$(O, o, 0 \mid -2, (2, 1), (2, 1), (2, 1), (2, 1)),$$

$$(O, o, 1 \mid 0),$$

$$(N, o, 1 \mid \epsilon),$$

$$(O, n, 1 \mid -1, (2, 1), (2, 1)),$$

$$(N, n, I, 1 \mid -, (2, 1), (2, 1)),$$

$$(O, n, 2 \mid 0),$$

$$(N, n, I-II, 2 \mid \epsilon)$$

where ϵ is either 0 or 1. The manifolds with boundary are the unique space with orbit surface a disk and two exceptional fibers of index 2, and the four spaces (orientable and non-orientable) with no exceptional fibers with orbit surface either an annulus or a Möbius band.

Recall that certain Seifert fibered spaces were identified in 2.3.3 as “small.” These are the spaces with finite Fuchsian quotient. Thus the hypotheses in the next statement require that the space be neither “small” nor “flat.”

LEMMA 2.4.27. *Let M be a compact, connected Seifert fibered space, let $h \in \pi_1(M)$ be represented by an ordinary fiber and assume that $\pi_1(M)/\langle h \rangle$ is infinite. Further assume that M is not “flat.” Then any cyclic normal subgroup of $\pi_1(M)$ is contained in $\langle h \rangle$.*

PROOF If the conclusion is false, then the hypotheses of Lemma 2.4.20 are satisfied and M is one of the spaces listed in the conclusion of that lemma. We also know that $\pi_1(M)$ contains $\mathbf{Z} \oplus \mathbf{Z}$ or $\mathbf{Z} \oplus \mathbf{Z} \oplus \mathbf{Z}$ as a subgroup N of finite index as described in Lemma 2.4.19.

We note that the “flat” Seifert fibered spaces are some of the spaces in the conclusion of Lemma 2.4.20. We must explain why not all of the spaces in the conclusion of Lemma 2.4.20 are listed among the “flat” spaces.

All of the spaces with boundary in the conclusion of Lemma 2.4.20 are listed as “flat.” All non-orientable spaces in the conclusion of Lemma 2.4.20 are listed as “flat.” The remaining spaces in Lemma 2.4.20 only show up among the “flat” spaces with specific values for the obstruction to a section. We must show why these are the only possible values under the hypotheses of the present lemma.

We assume that M is closed and orientable. Let N be the subgroup of finite index guaranteed by Lemma 2.4.19, and let \tilde{M} be the cover of M corresponding to N . We first argue that N is $\mathbf{Z} \oplus \mathbf{Z} \oplus \mathbf{Z}$. If not, then N is $\mathbf{Z} \oplus \mathbf{Z}$. Since N is of finite index, and M is closed, \tilde{M} is also closed. Since M is not “small,” it is aspherical as is \tilde{M} . This would make a closed 3-manifold have the homotopy type of $S^1 \times S^1$ which is impossible. Thus N is $\mathbf{Z} \oplus \mathbf{Z} \oplus \mathbf{Z}$.

We let $h \in \pi_1(M)$ be represented by an ordinary fiber. Lemma 2.4.19 says that $\langle h \rangle$ is in N , so \tilde{M} is a primary cover of M . We can regard $\pi_1(\tilde{M})$ as a subgroup of $\pi_1(M)$ and thus regard h as an element of $\pi_1(\tilde{M})$. We know that h is non-trivial. (In fact it has infinite order.) Thus $\pi_1(\tilde{M})/\langle h \rangle$ is a proper quotient of $\mathbf{Z} \oplus \mathbf{Z} \oplus \mathbf{Z}$. Since $\mathbf{Z} \oplus \mathbf{Z}_n$ is not a Fuchsian group, the quotient must be exactly $\mathbf{Z} \oplus \mathbf{Z}$.

From Lemma 2.4.5, we know that the orbit surface of \tilde{M} is a torus. Since $\pi_1(\tilde{M})$ is torsion free, we know that \tilde{M} has no exceptional fibers. The only information needed to determine \tilde{M} is \tilde{b} the obstruction to a section. A presentation for $\pi_1(\tilde{M})$ is $\langle h, x, y \mid x^{-1}y^{-1}xyh^{\tilde{b}} = 1 \rangle$. Since h has infinite order in $\pi_1(\tilde{M})$ we get that a and b commute only if $\tilde{b} = 0$. This makes \tilde{M} homeomorphic to $S^1 \times S^1 \times S^1$.

We now consider the structure of M . If M has n exceptional fibers, we let $(\mu_1, \beta_1), \dots, (\mu_n, \beta_n)$ be the crossing invariants of the exceptional fibers. Let

b be the obstruction to a section. From the previous section we know that $b + \sum(\mu_i/\beta_i)$ must be a divisor of \tilde{b} and therefore must be zero. Thus each space has $b = -\sum(\mu_i/\beta_i)$. This accounts for the list of closed orientable spaces that appear among the “flat” Seifert fibered spaces. ■

We take up the question of the uniqueness of the fiber structure of a Seifert fibered space. We restrict ourselves to Seifert fibered spaces with boundary. This vastly simplifies the arguments.

LEMMA 2.4.28. *Let M be a compact, connected 3-manifold with non-empty boundary with M neither a solid torus nor an I -bundle over a torus or Klein bottle. Let C be a component of ∂M and let J and K be simple closed curves in C that are fibers in two Seifert fiberings of M . Then J and K are isotopic in C .*

PROOF A fibered solid torus is the only “small” Seifert fibered space with non-empty boundary. The remaining spaces excluded by the hypothesis are the bounded “flat” Seifert fibered spaces. Note that the Seifert fibered space with orbit surface a disk and two exceptional fibers of index 2 is homeomorphic to an orientable I -bundle over a Klein bottle.

By Lemma 2.4.27, J and K generate the maximal cyclic normal subgroup of $\pi_1(M)$. Thus (up to reversal) they are homotopic in M . By Lemma 2.4.7, J and K are homotopic in C to powers of each other. Since $\pi_1(C)$ is $\mathbf{Z} \oplus \mathbf{Z}$, J and K are homotopic in C and thus isotopic in C . The isotopy can be extended to all of M . ■

Before we go further, we need some techniques to deal with incompressible surfaces in a Seifert fibered space. We start with a description of some canonical incompressible surfaces in a Seifert fibered space with boundary.

Let M be a compact, connected Seifert fibered space with non-empty boundary and assume that M is not a fibered solid torus. Let $p : M \rightarrow G$ be the

projection to the orbit surface. We know that either G is not a disk, or G has more than one exceptional point. There is a finite set $\{\alpha_1, \dots, \alpha_n\}$ of pairwise disjoint arcs properly embedded in G minus the exceptional points so that when G is split along the α_i , then each component is a disk, no component has more than one exceptional point, and no more than one component has no exceptional point. We call the saturated annuli $A_i = p^{-1}(\alpha_i)$ a "canonical system" of saturated annuli in M . Splitting M along the A_i represents M as a union of fibered solid tori and no more than one ordinary solid torus sewn together along saturated annuli in their boundaries.

LEMMA 2.4.29. *Let M be a fibered solid torus and let A be a properly embedded annulus in M . Assume that one boundary component H of A is a fiber in ∂M . Then there is an ambient isotopy of M rel H that carries A to a saturated annulus in M . If X is a union of saturated annuli in ∂M that is disjoint from $(\partial A) - H$, then the isotopy can be taken rel X . If both components of A are fibers in ∂M , then the isotopy can be taken rel ∂M .*

PROOF A fiber is not trivial in M . Thus the boundary components of A are parallel non-trivial curves in ∂M . If H' is the component of ∂A that is not H and H' is not a fiber, then there is an isotopy taking H' to a fiber. The isotopy can be taken rel H and rel X if X is as specified in the hypothesis. All further isotopies will be rel ∂M .

Since a fiber is not trivial in M , we know that A is incompressible in M . This expresses $\pi_1(M)$ as a free product with amalgamation over \mathbf{Z} , or an HNN extension over \mathbf{Z} . An HNN extension over \mathbf{Z} cannot yield \mathbf{Z} which is $\pi_1(M)$. Also, since $\pi_1(M)$ is \mathbf{Z} , the amalgamating subgroup must surject onto one of the factors. Thus one of the components T of M split along A is a product and A is boundary parallel. We have that T is a solid torus with boundary made of two

annuli A and A' with A' in ∂M . Let A'' be a saturated annulus parallel to A' and properly embedded in T with $\partial A'' = \partial A' = \partial A$. Now A and A'' are parallel and the result follows by creating an isotopy carrying A to A'' . This isotopy will be rel ∂M . ■

LEMMA 2.4.30. *Let M be a compact, connected Seifert fibered space with non-empty boundary and let A be a properly embedded annulus in M . Assume that one boundary component H of A is a fiber of ∂M . Then there is an ambient isotopy of M rel H that carries A to a saturated annulus in M . If X is a union of saturated annuli in ∂M that is disjoint from $(\partial A) - H$, then the isotopy can be taken rel X . If both components of ∂A are fibers, then the isotopy can be taken rel ∂M .*

PROOF If M is a fibered solid torus, then we are done by the previous lemma so we assume that it is not. If the boundary component H' of A that is not H is not a fiber then it is homotopic to a fiber. By Lemma 2.4.7, H' is homotopic in ∂M to a fiber in ∂M . (Here we use the fact that the only “small” Seifert fibered space with boundary is a fibered solid torus.) Since H' is a simple closed curve, it is isotopic to a fiber in ∂M . If H' is in the same component of ∂M as H , then we can assure that the isotopy does not disturb H . If X is as specified in the hypothesis, then we can also assure that the isotopy does not disturb X .

From now on, all isotopies will be rel ∂M . Since M is not a fibered solid torus, it has a canonical system $\{A_1, \dots, A_n\}$ of saturated annuli. Since both components of ∂A are fibers, we can assume that neither component of ∂A is in any of the A_i . We can isotope A to make the intersections of A with the A_i a set of simple closed curves that are non-trivial in A and the A_i . These are isotopic to fibers in the A_i and so we can isotope A so that the intersections are fibers. We now use the previous lemma to fix up the parts of A that remain when we

split M along the A_i . The fix will be done relative to the boundaries of the pieces of M that are created by the splitting. ■

LEMMA 2.4.31. *Let M and N be compact, connected Seifert fibered spaces with boundary and let $f : M \rightarrow N$ be a homeomorphism that takes a fiber H in ∂M to a fiber in ∂N . Then f is isotopic (rel H) to a fiber preserving homeomorphism.*

PROOF If the orbit surface of one is a disk with no more than one exceptional point, then both M and N are fibered solid tori. The hypotheses say that they have the same invariants and have the same fiber type. An isotopy can be built in a manner similar to previous exercises.

We now assume that G , the orbit surface of M is not a disk or it contains at least two exceptional fibers. Let α be a properly embedded arc in G that misses all exceptional points, that has the image of H in G as one of its boundary points, and so that α is not boundary parallel in G minus the exceptional points. Then the pre-image of α in M is a saturated annulus A containing H as one boundary component. The image of A in N is an annulus that is properly embedded in N with one boundary component a fiber of N . By the previous lemma, there is an ambient isotopy of N carrying the image of A to a saturated annulus in N . Thus we may assume that the image of A is a saturated annulus in N . If we split M along A and N along the image of A , then we obtain spaces with “simpler” orbit surfaces and can repeat the process. We can assure that the annuli used for later steps are disjoint from the copies of A and that the isotopies do not disturb the copies of A .

We can do this for each element of a canonical system of M . At the end we are left with fibered solid tori with the map already fiber preserving along certain annuli in the boundary. The lemma is finished by fixing the maps on these fibered solid tori rel the annuli. This is similar to previous exercises. ■

THEOREM 2.4.32. *Let M and N be compact, connected Seifert fibered spaces with $f : M \rightarrow N$ a homeomorphism. Assume that M has non-empty boundary and is homeomorphic to neither a solid torus nor an I -bundle over a torus or Klein bottle. Then f is isotopic to a fiber preserving homeomorphism.*

PROOF By Lemma 2.4.28, a fiber in ∂M has image that is isotopic to a fiber in ∂N . Thus we can assume that f takes a fiber in ∂M to a fiber in ∂N . We now apply Lemma 2.4.31. ■